Question 5 (a).

A set of closed formulas $\Gamma$ is said to be maximal if for every closed formula $A$, either $A \in \Gamma$ or $\neg A \in \Gamma$. Show that if $\Gamma$ is maximal and consistent, then

(a) $A \lor B \in \Gamma$ if and only if $A \in \Gamma$ or $B \in \Gamma$.

(a) $\Rightarrow$: Assume $A \lor B \in \Gamma$. We show $A \in \Gamma$ or $B \in \Gamma$. Assume, for contradiction, this is not the case, that is, neither $A \in \Gamma$ nor $B \in \Gamma$. By maximality of $\Gamma$ it follows that $\neg A \in \Gamma$ and $\neg B \in \Gamma$. Now we have the following derivation of $\bot$ from assumptions in $\Gamma$:

\[
\begin{align*}
  u : A \lor B & \quad v : A \rightarrow \bot & \quad w : B \rightarrow \bot \\
  \bot & \quad \lor & \\
\end{align*}
\]

This contradicts the consistency of $\Gamma$.

$\Leftarrow$: Assume $A \in \Gamma$ or $B \in \Gamma$. W.l.o.g. (Without loss of generality) we assume $A \in \Gamma$. We show $A \lor B \in \Gamma$. Assume, for contradiction, this is not the case, that is, $A \lor B$ is not in $\Gamma$. By maximality of $\Gamma$ it follows that $\neg (A \lor B) \in \Gamma$. Now we have the following derivation of $\bot$ from assumptions in $\Gamma$:

\[
\begin{align*}
  v : (A \lor B) \rightarrow \bot & \quad u : A \lor B \\vdash \bot \\
  \bot & \quad \lor & \\
\end{align*}
\]

Again, this contradicts the consistency of $\Gamma$. 