Solutions to Coursework 2

Question 1.

Derive the formula $\neg (A \leftrightarrow \neg A)$ in minimal logic. You may construct the derivation by hand, or use the interactive proving tool from the logic labs.

We use the abbreviation $B := (A \rightarrow \neg A) \land (\neg A \rightarrow A)$

A direct proof of $\neg B$ goes as follows:

\[
\frac{u : B}{A \rightarrow \neg A} \land \neg 1 \\
\frac{\neg A}{\neg A} \land \neg r \\
\frac{A}{A} \rightarrow \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{u : B}{A \rightarrow \neg A} \land \neg 1 \\
\frac{\neg A}{\neg A} \land \neg r \\
\frac{w : A}{w : A} \rightarrow \\
\frac{v : A}{v : A} \rightarrow \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{u : B}{A \rightarrow \neg A} \land \neg 1 \\
\frac{\neg A}{\neg A} \land \neg r \\
\frac{w : A}{w : A} \rightarrow \\
\frac{v : A}{v : A} \rightarrow \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow +
\]

The proof above contains several identical subproof. Using some clever “detours” or “lemmas” one can avoid repeated occurrences of identical sub proofs:

\[
\frac{v : A}{v : A} \rightarrow \\
\frac{\neg A}{\neg A} \land \neg r \\
\frac{w : A}{w : A} \rightarrow \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{u : B}{A \rightarrow \neg A} \land \neg 1 \\
\frac{\neg A}{\neg A} \land \neg r \\
\frac{w : A}{w : A} \rightarrow \\
\frac{v : A}{v : A} \rightarrow \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow + \\
\frac{\bot}{\bot} \rightarrow +
\]

Question 2.

Define suitable signatures for the following statements and formalise them.

(a) If anyone in the dorm has the measles, then everyone who has a friend in the dorm will have to be quarantined.

(b) If anyone can do it, Jones can.

(c) If Jones can do it, anyone can.

In all cases there is only one sort, let’s call it $p$, for “person”. In part (c) there is a constant $J$. Otherwise there are no constants or function symbols.

Hence, in order to determine the signatures, it suffices in each case to define the predicate symbols and their arities.

(a) Signature: $D : (p)$ (lives in dorm), $M : (p)$ (has the measles), $Q : (p)$ (has to be quarantined), $F : (p \times p)$ (are friends).

\[
\exists x (D(x) \land M(x)) \rightarrow \forall x (\exists y (F(x,y) \land D(y)) \rightarrow Q(x)).
\]

(b) Signature: $D : (p)$ (can do it).

\[
\exists x D(x) \rightarrow D(J) \quad \text{(equivalent solution } \forall x (D(x) \rightarrow D(J))\text{)}.
\]
(c) Signature as in (b).
\[ D(J) \rightarrow \forall x \, D(x) \] (equivalent solution \( \forall x \, (D(J) \rightarrow D(x)) \)).

Question 3.

Define a suitable signature of the following statements and formalise them. What are the free variables in each statement?

(a) Every number that is larger than \( x \) is larger than \( y \).
(b) For every number \( a \), the equation \( ax^2 + 4x - 2 = 0 \) has exactly one solution iff \( a \geq -2 \).

In both cases there is only one sort, let’s call it \( r \), for “real numbers”.

(a) Signature: \( >; (r \times r) \) (greater than).
\[ \forall z \,(z > x \rightarrow z > y) \]. The free variables are \( x \) and \( y \).

(b) Signature:
Constants: 0, 2, 4: \( r \).
Function symbols: \(-: r \rightarrow r\) (negation), \(+, \cdot: r \times r \rightarrow r\) (addition, multiplication).
Predicate symbols: \( \geq: (r \times r) \) (greater or equal).

We use the abbreviations \( x^2 := x \times x, \ xy := x \times y, \ x - y := x + (-y), \) and \( x + y + z := x + (y + z) \).
The required formula is \( \forall a \,(\exists_1 x \,(ax^2 + 4x - 2 = 0) \leftrightarrow a \geq (-2)) \)
where \( \exists_1 y A(x) \) is shorthand for \( \exists x \,(A(x) \land \forall y \,(A(y) \rightarrow x = y)) \).
In our example \( A(x) \) is the formula \( ax^2 + 4x - 2 = 0 \). \( \exists_1 y A(x) \) could equivalently be defined as \( \exists x \forall y \,(A(y) \leftrightarrow y = x) \).
The formula \( \forall a \,(\exists_1 x \,(ax^2 + 4x - 2 = 0) \leftrightarrow a \geq (-2)) \) is closed, that is, it has no free variables.

Question 4.

Abbreviations: \( N(x) := \neg \exists u \, R(x, u) \) (\( x \) is in normal form), \( S(x) := \forall u \, R(u, x) \) (\( x \) is a sink).

(a) \( W := \forall x \,(\neg N(x) \rightarrow \exists y \, (R(x, y) \land N(y))) \).
(b) \( W \rightarrow \neg \exists x \, S(x) \).

Question 5. We have to show \( W \rightarrow \neg \exists x \, S(x) \). Informal argument: Assume \( W \) and assume there exists a sink. We have to show \( \bot \). Let \( x \) be a sink (use \( \exists^+ \) to get such an \( x \)). Then \( R(x, x) \), i.e. \( x \) is not in normal form. Hence, by assumption \( W \), \( \exists y \,(R(x, y) \land N(y)) \). Let \( y \) be such that \( R(x, y) \) and \( N(y) \) (use \( \exists^- \) to get such a \( y \)). Because \( x \) is a sink it follows \( R(y, x) \), but this contradicts the normality of \( y \).

Formal proof:

\[
\begin{align*}
& w: W \\
\implies & \neg N(x) \rightarrow \exists y \,(R(x, y) \land N(y)) \\
\implies & \neg N(x) \\
\implies & \exists y \,(R(x, y) \land N(y)) \\
\implies & \exists y \,(R(x, y) \land N(y)) \\
\implies & n: N(x) \\
\implies & \exists z : S(x) \\
\implies & \forall x \, S(x) \\
\implies & W \rightarrow \neg \exists x \, S(x) \\
& w: W \\
\end{align*}
\]