The Thue-Morse Sequence

By a binary word, or word for short, we mean a finite list of zeros and ones. Following common practice, we denote a word by simple writing its elements next to each other, for example, 011001. The empty word is denoted by \( \varepsilon \), and \( vw \) denotes the concatenation of the words \( v \) and \( w \).

Note that in Haskell the word 011001 would be implemented as \([0, 1, 1, 0, 0, 1]\).

Consider the following operations on words:

\[
\begin{align*}
\tau(w) & = \text{the result of replacing in } w \text{ 0 by 01 and 1 by 10} \\
\nu(w) & = \text{the result of replacing in } w \text{ 0 by 1 and 1 by 0}
\end{align*}
\]

For example, \( \tau(011) = 011010 \) and \( \nu(011) = 100 \).

In the following you may use without proof the following facts:

(a) \( \tau(vw) = \tau(v)\tau(w) \),

(b) \( \tau(\nu(w)) = \nu(\tau(w)) \).

For every natural number \( n \) we define inductively words \( v_n \) and \( w_n \) as follows

\[
\begin{align*}
v_0 & = 0 \\
v_{n+1} & = \tau(v_n) \\
w_0 & = 0 \\
w_{n+1} & = w_n\nu(w_n)
\end{align*}
\]

Question 1. Implement two Haskell functions computing \( v_n \) respectively \( w_n \) for every \( n \). Confirm empirically that \( v_n = w_n \) holds for all \( n \).

Question 2. Prove \( v_n = w_n \) for all \( n \).

Hint: Show first \( \tau(v_n) = v_n\nu(v_n) \) for all \( n \).