Question 1. Let $f: \mathbb{N} \to \mathbb{N}$ be a partial recursive function such that for every $y \in \mathbb{N}$ there exists $x \in \text{dom}(f)$ with $f(x) \simeq y$. We have to construct a total recursive function $g: \mathbb{N} \to \mathbb{N}$ such that for all $y \in \mathbb{N}$ we have $g(y) \in \text{dom}(f)$ and $f(g(y)) \simeq y$. To this end, we apply Kleene’s Normal Form Theorem to $f$. Let $e \in \mathbb{N}$ be an index for $f$, that is, $f = \{e\}^{(1)}$. Then for all $x \in \mathbb{N}$
\[ f(x) \simeq U(\mu z. T_1(e, x, z)) \]
Define a function $h: \mathbb{N} \to \mathbb{N}$ by
\[ h(y) := \mu p_1[T_1(e, \pi_1(p), \pi_2(p)) \land \forall z < \pi_2(p) \rightarrow T_1(e, \pi_1(p), z) \land U(\pi_1(p)) = y] \]
By the assumption on $f$, the function $h$ is total, and, since the expression behind the quantifiers defines a decidable relation of $y$ and $p$, $f$ is computable. Obviously $f(\pi_1(h(y))) \simeq y$. Hence we define $g: \mathbb{N} \to \mathbb{N}$ by $g(y) := \pi_1(h(y))$.

Question 2.
(a) Set $K := \{ e \in \mathbb{N} \mid \{ e \}^{(1)}(e) \text{ is defined} \} \subseteq \mathbb{N}$. By Turing’s theorem on the existence of universal machines, the partial function $f: \mathbb{N} \to \mathbb{N}$, $f(e) := \{ e \}^{(1)}(e)$, is partial recursive. Since $K = \text{dom}(f)$, it follows that $K$ is recursively enumerable.

We show the undecidability of $K$ by a proof similar to the proof of the undecidability of the Halting Problem. Before we do that we note that for any URM $Q$ and $x \in \mathbb{N}$ we have $\{ \text{code}(Q) \}^{(1)}(x) \simeq Q^{(1)}(x)$ and therefore
\[ \text{code}(Q) \in K \Leftrightarrow \{ \text{code}(Q) \}^{(1)}(\text{code}(Q)) \text{ is defined} \Leftrightarrow Q^{(1)}(\text{code}(Q)) \text{ is defined} \]
Proof of the undecidability of $K$: Assume there is a program $P$ deciding $K$, that is, for all $e \in \mathbb{N}$
\[ P^{(1)}(e) = \begin{cases} 1 & \text{if } e \in K \\ 0 & \text{otherwise} \end{cases} \]
We show that this assumption must be false.

Using our assumed program $P$ we can easily construct another URM program $R$ which has the property that for all $e \in \mathbb{N}$
\[ R^{(1)}(e) = \begin{cases} 0 & \text{if } P^{(1)}(e) = 0 \\ \text{undefined} & \text{if } P^{(1)}(e) = 1 \end{cases} \]
From the construction of $R$ it follows that for every URM program $Q$ the following equivalences hold:
\[ R^{(1)}(\text{code}(Q)) \text{ is defined} \Leftrightarrow P^{(1)}(\text{code}(Q)) = 0 \Leftrightarrow \text{code}(Q) \notin K \]
Since, by the remark above $\text{code}(Q) \in K$ if and only if $Q^{(1)}(\text{code}(Q))$ is defined, we have
\[ (+) \quad R^{(1)}(\text{code}(Q)) \text{ is defined} \Leftrightarrow Q^{(1)}(\text{code}(Q)) \text{ is undefined} \]
Since $(+)$ holds for all URM programs $Q$, it particularly holds for $Q := R$. This gives us
\[ R^{(1)}(\text{code}(R)) \text{ is defined} \Leftrightarrow R^{(1)}(\text{code}(R)) \text{ is undefined} \]
which is absurd.

(b) Obviously, $e \in K$ if and only if $(e, e) \in \text{Halt}$, hence $K$ is reducible to $\text{Halt}$ via the recursive function $h: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $h(e) := (e, e)$. Therefore, the undecidability of $K$ implies the undecidability of $\text{Halt}$.

p.t.o.
Question 3. Let $f: \mathbb{N} \to \mathbb{N}$ be a partial recursive function. In order to see that the set $A := \{ f(x) \mid x \in \text{dom}(f) \}$ is recursively enumerable, we apply Kleene's Normal Form Theorem once more. Let $e \in \mathbb{N}$ such that for all $x \in \mathbb{N}$ $f(x) \simeq U(\mu z. T_1(e, x, z))$ We have

$$A = \{ y \in \mathbb{N} \mid \exists x (x \in \text{dom}f \land f(x) \simeq y) \}
= \{ y \in \mathbb{N} \mid \exists x \exists z [T_1(e, x, z) \land \forall z'< z \sim T_1(e, x, z') \land U(z) = y] \}
$$

Since the expression in square brackets defines a decidable condition on $y, x$ and $z$, $A$ is recursively enumerable, according to the first characterization theorem for r.e. predicates.

Question 4.

(a) Let $A \subseteq \mathbb{N}$. Obviously, $A \in \text{co}-\Pi^0_1 \iff A \in \Pi^0_2$ (since $\mathbb{N} \setminus (\mathbb{N} \setminus A) = A$). Because for any formula $F(x)$, we have

$$(*) \quad \neg \forall x F(x) \iff \exists x \neg F(x) \quad \text{and} \quad \neg \exists x F(x) \iff \forall x \neg F(x)
$$

it follows that for any set $A \subseteq \mathbb{N}$:

$$A \in \text{co}-\Pi^0_1
\iff \mathbb{N} \setminus A \in \Pi^0_2
\iff \mathbb{N} \setminus A = \{ x \in \mathbb{N} \mid \forall x_1 \exists x_2 \ldots R(x_1, x_2, \ldots, x_n, x) \} \text{ for some recursive predicate } R
\iff A = \{ x \in \mathbb{N} \mid \neg \forall x_1 \exists x_2 \ldots R(x_1, x_2, \ldots, x_n, x) \} \text{ for some recursive predicate } R$$

$$(*) \quad A = \{ x \in \mathbb{N} \mid \exists x_1 \forall x_2 \ldots \neg R(x_1, x_2, \ldots, x_n, x) \} \text{ for some recursive predicate } R$$

$$(** \quad A = \{ x \in \mathbb{N} \mid \exists x_1 \forall x_2 \ldots S(x_1, x_2, \ldots, x_n, x) \} \text{ for some recursive predicate } S
\iff A \in \Sigma^0_n
$$

Equivalence $(**)$ holds because the recursive predicates are closed under complements.

(b) By the first characterization theorem for r.e. predicates, a set is recursively enumerable if and only if it is in $\Sigma^0_1$. By the characterization theorem for decidable predicates, a set is decidable if and only if itself and its complement are recursively enumerable. Together we obtain

$$A \text{ decidable if and only if } A \in \Sigma^0_1 \text{ and } \mathbb{N} \setminus A \in \Sigma^0_1.
$$

By (a) this means

$$A \text{ decidable if and only if } A \in \Sigma^0_1 \text{ and } \mathbb{N} \setminus A \in \text{co}-\Pi^0_1,$$

that is,

$$A \text{ decidable if and only if } A \in \Sigma^0_1 \text{ and } A \in \Pi^0_1.
$$

Since $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$, it follows

$$A \text{ decidable if and only if } A \in \Delta^0_1
$$

Question 5. Let $A := \{ e \in \mathbb{N} \mid \{3, 4\} \subseteq \text{dom}(\{e\}(1))\}.$

(a) According to Turing’s theorem on the existence of universal machines, the partial functions $f: \mathbb{N} \to \mathbb{N}$, $f(e) := \{e\}(1)(3)$ and $g: \mathbb{N} \to \mathbb{N}$, $g(e) := \{e\}(1)(4)$ are partial recursive. Obviously, $A = \text{dom}(f) \cap \text{dom}(g)$, therefore, $A$ is recursively enumerable, because, according to the lecture, the r.e. sets are closed under intersection.

(b) Set $\mathcal{F} := \{ f: \mathbb{N} \to \mathbb{N} \mid f \text{ partial recursive and } \{3, 4\} \subseteq \text{dom}(f) \}$. Then $A$ is the set of Gödel numbers of $\mathcal{F}$. Because $\mathcal{F}$ is proper (the zero function is in $\mathcal{F}$, the nowhere defined function is not in $\mathcal{F}$), it follows, by Rice’s Theorem, that $A$ is undecidable.

(c) If $\mathbb{N} \setminus A$ were recursively enumerable, then, by (a) and the characterization theorem for decidable predicates, $A$ would be decidable, contradicting (b).