Question 1. Let \( f : A \rightarrow B \) be a function. Define \( g : \mathcal{P}(A) \rightarrow \mathcal{P}(B) \) by
\[
g(X) := f[X] := \{ f(a) | a \in X \}
\]
for every \( X \subseteq A \).
Prove that \( f \) is injective if and only if \( g \) is injective. \( \text{[5 marks]} \)

Question 2. Prove that \( \Sigma_{i \leq n} 1 + 4i = 2n^2 + 3n + 1 \) for all \( n \in \mathbb{N} \). \( \text{[4 marks]} \)

Question 3. Prove or disprove the following statements:
(a) For every set \( A \) the powerset, \( \mathcal{P}(A) \), is either finite or uncountable.
(b) For all sets \( A, B \) the sets \( A \rightarrow B \) and \( B \rightarrow A \) have the same cardinality. \( \text{[6 marks]} \)

Question 4. Write a URM-program for the function \( f : \mathbb{N} \rightarrow \mathbb{N} \), \( f(x) := 2x \). \( \text{[5 marks]} \)

Question 5. Show that the factorial function, \( x! \), is primitive recursive. \( \text{[3 marks]} \)

Question 6. Let \( f : \mathbb{N}^2 \rightarrow \mathbb{N} \) be a function satisfying for all \( x, y \in \mathbb{N} \) the equations
\[
f(x, 0) = x
\]
\[
f(x, y + 1) = f(x!, y)
\]
(a) Prove that \( f(x!, y) = f(x, y)! \) for all \( y, x \in \mathbb{N} \).
(b) Use (a) and Question 4 to show that \( f \) is primitive recursive. \( \text{[7 marks]} \)

Question 7. In the following we only consider URMs with a fixed number of registers, say 10 registers. A total universal URM is a URM \( U \) such that \( U^{(2)} \) is total and that is able to simulate any total URM in the following sense. Given a URM \( P \) such that \( P^{(1)} \) is total, there is a number \( e \) such that for all \( x \) we have \( P^{(1)}(x) = U^{(2)}(x, e) \).
Prove that a total universal URM does not exist. \( \text{[10 marks]} \)

Question 8. Compute the normal form of \( c_2c_2 \). \( \text{[5 marks]} \)

Question 9. Use the Church-Rosser property of \( \beta \)-reduction to show that the relation \( M =_\beta N \) is an equivalence relation on the set of all lambda-terms. \( \text{[5 marks]} \)