Solutions to Coursework 2

Question 1. (a) We prove \( t^{T(\Sigma)} = t \) for every closed \( \Sigma \)-term \( t \) by structural induction on \( t \).

Base. \( c^{T(\Sigma)} = c \), by definition.

Step. 
\[
f(t_1, \ldots, t_n)^{T(\Sigma)} = f^{T(\Sigma)}(t_1^{T(\Sigma)}, \ldots, t_n^{T(\Sigma)}) \\
= f(t_1^{T(\Sigma)}, \ldots, t_n^{T(\Sigma)}) \quad \text{by definition of } f^{T(\Sigma)} \\
= f(t_1, \ldots, t_n) \quad \text{by induction hypothesis}
\]

(b) We prove \( f^{T(\Sigma), \theta} = 	heta(t) \) for every term \( t \in T(\Sigma, X) \) and every substitution \( \theta : X \rightarrow T(\Sigma, Y) \) by structural induction on \( t \).

Base: constants. \( c^{T(\Sigma), \theta} = c = c \theta \).

Base: variables. \( x^{T(\Sigma), \theta} = \theta(x) = x \theta \).

Step. 
\[
f(t_1, \ldots, t_n)^{T(\Sigma), \theta} = f^{T(\Sigma), \theta}(t_1^{T(\Sigma), \theta}, \ldots, t_n^{T(\Sigma), \theta}) \\
= f(t_1^{T(\Sigma), \theta}, \ldots, t_n^{T(\Sigma), \theta}) \quad \text{by definition of } f^{T(\Sigma), \theta} \\
= f(t_1, \theta \circ \theta, \ldots, t_n \theta) \quad \text{by induction hypothesis} \\
= f(t, \ldots, t \theta)
\]

Question 2. The general assumptions are that \( A, B \in C \) and \( A \) is initial in \( C \). We have to show 

\[ B \text{ is initial for } C \iff B \cong A \]

\[ \text{"} \implies \text{"} \]: Assume \( B \) is initial for \( C \). Because \( A, B \in C \) and both are initial for \( C \), there are homomorphisms \( \varphi_1 : A \rightarrow B \) and \( \varphi_2 : A \rightarrow B \). For proving \( B \cong A \) it suffices to show that \( \varphi_1 \) and \( \varphi_2 \) are inverses of each other. Since homomorphisms are closed under composition we know that \( \varphi_2 \circ \varphi_1 : A \rightarrow A \) is a homomorphism. Furthermore the identity \( \text{id}_A : A \rightarrow A \) is a homomorphism. By initiality of \( A \) we must have \( \varphi_2 \circ \varphi_1 = \text{id}_A \). Similarly one shows that \( \varphi_1 \circ \varphi_2 = \text{id}_B \).

\[ \text{"} \iff \text{"} \]: Assume \( B \cong A \). Let \( \varphi : B \rightarrow A \) be an isomorphism. In order to prove that \( B \) is initial for \( C \) we choose an arbitrary \( \Sigma \)-algebra \( C \in C \) and have to show that there is a unique homomorphism from \( B \) to \( C \). Since \( A \) is initial for \( C \) there is a homomorphism \( \psi : A \rightarrow C \). Hence we have the homomorphism \( \psi \circ \varphi : B \rightarrow C \). In order to show that this homomorphism is unique we take an arbitrary homomorphism \( \chi : B \rightarrow C \) and have to prove that \( \chi = \psi \circ \varphi \). Composing \( \chi \) with the inverse of \( \varphi, \varphi^{-1} : A \rightarrow B \), we obtain the homomorphism \( \chi \circ \varphi^{-1} : A \rightarrow C \). Since \( A \) is initial for \( C \) we must have \( \chi \circ \varphi^{-1} = \psi \). Composing both sides of this equation to the right with \( \varphi \) we see that \( \chi = \psi \circ \varphi \).

p.t.o.
In the intended algebra $A$ of natural numbers and integers the operation make-int computes the difference of two natural numbers, and all other constants and operations are interpreted in the expected way.

In order to show that $A$ is indeed a model of the initial specification $NATINT$ (or $NATINT$ is adequate for $A$) we show (according to Theorem 3.4.4(v))

(i) The equations are correct, i.e. their universal closures hold in $A$. This is obviously true.

(ii) $A$ is generated, i.e. every element of a carrier set in $A$ is the value of a closed term. That’s also obvious: every natural number $n$ is the value of $\text{succ}^n(0)$, every nonnegative integer $i$ is the value of $\text{make-int}(\text{succ}^{|i|}(0), 0)$ or $\text{make-int}(0, \text{succ}^{|i|}(0))$ depending on whether $i$ is non-negative or negative.

(iii) The set $E$ of equations of $NATINT$ is complete for $A$, i.e. for any closed terms $t_1, t_2$ that have the same value in $A$ (i.e. $A \models t_1 = t_2$) the equation $t_1 = t_2$ follows from $E$ (i.e. $\forall E \models t_1 = t_2$), or, equivalently, using Birkhoff’s Theorem, $E \vdash t_1 = t_2$. If $t_1, t_2$ are both of sort nat this is trivial, since the only closed terms of sort nat are the terms $\text{succ}^n(0)$ and hence $t_1, t_2$ have the same value only if they are identical. Furthermore one can easily prove for all closed terms $t$ of sort int that $E \vdash t = \text{make-int}(\text{succ}^n(0), 0)$ or $E \vdash t = \text{make-int}(0, \text{succ}^n(0))$, where $n = |t|^4$, depending on the sign of $t$. Hence it suffices to consider terms of the form $\text{make-int}(\text{succ}^n(0), 0)$ and $\text{make-int}(0, \text{succ}^n(0))$. But, again, those terms have the same value only if they are identical.

Remarks. 1. The initial specification above uses only the constant 0 and the operation succ as auxiliaries. There are other (simpler) solutions that use more auxiliary operations (e.g. addition and multiplication on the natural numbers, which, of course, then have to be specified by equations as well).

2. The proof that the produced initial specification is indeed adequate was not part of the question. It has been added here to exemplify the criteria of adequacy for initial specifications.
Init Spec \[NATLIST\]

Sorts \[\text{nat, list, boole}\]

Constants \[0: \text{nat}, \text{nil}: \text{list}, \text{T: boole}, \text{F: boole}\]

Operations \[\text{succ: nat} \rightarrow \text{nat} \]
\[\text{cons: nat} \times \text{list} \rightarrow \text{list}\]
\[\text{length: list} \rightarrow \text{nat}\]
\[\leq: \text{nat} \times \text{nat} \rightarrow \text{boole}\]
\[\text{empty: list} \rightarrow \text{boole}\]
\[\text{ordered: list} \rightarrow \text{boole}\]
\[\text{equal: list} \times \text{list} \rightarrow \text{boole}\]
\[\land: \text{boole} \times \text{boole} \rightarrow \text{boole}\]
\[\lor: \text{boole} \times \text{boole} \rightarrow \text{boole}\]
\[\text{member: nat} \times \text{list} \rightarrow \text{boole}\]

Variables \[x,y: \text{nat}, \ l: \text{list}, \ b: \text{boole}\]

Equations \[\text{length}(\text{nil}) = 0\]
\[\text{length}(\text{cons}(x,l)) = \text{succ}(\text{length}(l))\]

\[\text{empty}(\text{nil}) = \text{T}\]
\[\text{empty}(\text{cons}(x,l)) = \text{F}\]

\[x \leq x = \text{T}\]
\[\text{succ}(x) \leq 0 = \text{F}\]
\[0 \leq \text{succ}(y) = \text{T}\]
\[\text{succ}(x) \leq \text{succ}(y) = x \leq y\]

\[\text{T} \land b = b\]
\[\text{F} \land b = \text{F}\]

\[\text{ordered}(\text{nil}) = \text{T}\]
\[\text{ordered}(\text{cons}(x,\text{nil})) = \text{T}\]
\[\text{ordered}(\text{cons}(x,\text{cons}(y,l))) = x \leq y \land \text{ordered}(\text{cons}(y,l))\]

\[\text{equal}(x,x) = \text{T}\]
\[\text{equal}(\text{succ}(x),0) = \text{F}\]
\[\text{equal}(0,\text{succ}(y)) = \text{F}\]
\[\text{equal}(\text{succ}(x),\text{succ}(y)) = \text{equal}(x,y)\]

\[\text{T} \lor b = \text{T}\]
\[\text{F} \lor b = b\]

\[\text{member}(x,\text{nil}) = \text{F}\]
\[\text{member}(x,\text{cons}(y,l)) = \text{equal}(x,y) \lor \text{member}(x,l)\]
The adequacy of the initial specification $NATLIST$ for the intended algebra is shown similarly as in question 3.

**Question 5.** It is fairly easy to see that the term rewriting systems associated with $NATINT$ and $NATLIST$ are both confluent and terminating. Hence rapid prototyping is possible in both cases.

The closed normal forms of $NATINT$ are the terms

\[ \text{succ}^n(0) \]
\[ \text{make-int}(\text{succ}^n(0), 0) \]
\[ \text{make-int}(0, \text{succ}^n(0)) \]

The closed normal forms of $NATLIST$ are the terms

\[ \text{succ}^n(0) \]
\[ T, \ F \]
\[ \text{cons}(\text{succ}^{n_1}(0), \ldots, \text{cons}(\text{succ}^{n_k}(0), \text{nil}) \ldots) \]