CS 376 Programming with Abstract Data Types

Coursework 1

**Question 1.** Consider the signature

<table>
<thead>
<tr>
<th>Signature</th>
<th>( \Sigma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sorts</td>
<td>nat</td>
</tr>
<tr>
<td>Constants</td>
<td>0: nat, 1: nat</td>
</tr>
<tr>
<td>Operations</td>
<td>( +: ) nat \times nat \rightarrow nat |</td>
</tr>
<tr>
<td></td>
<td>( \times: ) nat \times nat \rightarrow nat</td>
</tr>
</tbody>
</table>

Write down \( \Sigma \)-formulas expressing the following statements (in the \( \Sigma \)-algebra of natural number with 0, 1, addition and multiplication):

(a) \( x \) is a prime number. \[4 \text{ marks}\]

(b) Every prime number is the sum of four squares. \[4 \text{ marks}\]

(c) 2 is the only even prime number. \[4 \text{ marks}\]

You may use infix notation, and in (b) and (c) you may use a name, say \( \text{Prime}(x) \), for the formula found in (a).

**Question 2.** Let \( P: \equiv \forall y (y \neq x \rightarrow \exists x (x + 1 = y)) \), and \( t: \equiv y \times x \).

Compute the formula \( P\{t/x\} \). \[4 \text{ marks}\]

**Question 3.** Let \( \Sigma \) be a signature and \( \theta: Y \rightarrow T(\Sigma, X) \) a substitution. Note that \( \theta \) can also be viewed as a variable assignment in the term algebra \( T(\Sigma, X) \) (see definition 2.2.7 in the course notes). Prove by induction on terms that for any term \( t \in T(\Sigma, Y) \)

\[
i_{T(\Sigma, X), \theta}^{T(\Sigma, X)} = t_\theta
\]

[10 marks]

**Question 4.** Give natural deduction proofs of the following formulas.

(a) \( \neg(P \leftrightarrow \neg P) \) \[8 \text{ marks}\]

(b) \( (\neg P \lor Q) \rightarrow (P \rightarrow Q) \) \[8 \text{ marks}\]

(c) \( P \lor \neg P \) \[8 \text{ marks}\]

Hint: (a) is derivable in minimal logic, (b) in intuitionistic logic and (c) in classical logic.

**Date due:** 8 November 2001