Question 1. Let $\Sigma$ be the signature consisting of one sort $s$, a constant $c: s$ and one binary operation $f: s \times s \rightarrow s$. Consider the following $\Sigma$-algebras $A$, $B$, $C$:

$A_s := \mathbb{R} =$ the set of real numbers,
$c^A := 0$,
$f^A(r_1, r_2) := r_1 + r_2 \ (r_1, r_2 \in \mathbb{R})$, that is, $f^A$ is addition.

$B_s := \mathbb{R}_0^+ = \{ r \in \mathbb{R} \mid r \geq 0 \}$,
$c^B := 0$,
$f^B(r_1, r_2) := r_1 + r_2 \ (r_1, r_2 \in \mathbb{R}_0^+)$.

$C_s := \mathbb{R}^+ = \{ r \in \mathbb{R} \mid r > 0 \}$,
$c^C := 1$,
$f^C(r_1, r_2) := r_1 \ast r_2 \ (r_1, r_2 \in \mathbb{R}^+)$, that is, $f^C$ is multiplication.

For each of the algebras $B$ and $C$ decide whether or not it is isomorphic to the algebra $A$. Justify your answers. [20 marks]

Question 3.

(a) Produce an initial specification for the algebra of nonempty finite sets of natural numbers that has the following operations:

- singleton: creating a set with one element;
- union: the union of two sets;
- element: testing whether a number is an element of a set.
- card: the cardinality of a set;

You may import suitable data types of natural numbers and booleans, and add further operations. You may also use familiar mathematical notation. For example, you may write $\{x\}$ for singleton($x$), $s \cup t$ for union($s, t$), e.t.c. [30 marks]
(b) Determine a minimal set of generators in the specification you have given in (a) above. Are these generators free? [5 marks]

(c) Suppose you added an operation \texttt{count : set} \rightarrow \texttt{nat} and the following equations to your initial specification:

\begin{align*}
\text{count}(\{x\}) &= 1 \\
\text{count}(s \cup t) &= \text{count}(s) + \text{count}(t)
\end{align*}

Explain why the extended specification would be flawed. [5 marks]

**Question 4.** Let \( s \) be a sort, \( 1: s \) a constant, \(*, \texttt{exp}: \texttt{nat} \times s \rightarrow s\) operations (* used in infix notation), and \( x, y, z: s \) variables. Consider the term rewriting system \( R \) given by the following rules:

\begin{align*}
x * 1 &\rightarrow x \\
1 * x &\rightarrow x \\
\texttt{exp}(1, x) &\rightarrow 1 \\
\texttt{exp}(x, 1) &\rightarrow x \\
\texttt{exp}(x * y, z) &\rightarrow \texttt{exp}(x, z) * \texttt{exp}(y, z) \\
\texttt{exp}(x, y * z) &\rightarrow \texttt{exp}(\texttt{exp}(x, y), z)
\end{align*}

Prove that \( R \) is terminating. [40 marks]

**Date due: Tuesday, 8 December 2008**