CS_376 Programming with Abstract Data Types

Coursework 2

Question 1. Let $\Sigma$ be the signature with one sort, one constant and one binary operation. Let $A$ be the $\Sigma$-algebra of natural numbers with 0 and addition.

Which of the following relations are congruences on $A$? Justify your answers.

(a) $n \sim_1 m : \iff n \leq m$.
(b) $n \sim_2 m : \iff n \cdot m > 0$.
(c) $n \sim_3 m : \iff |n - m|$ is divisible by 3.

In case the relation is a congruence, describe the corresponding quotient algebra and state how many elements the carrier of the quotient algebra has.

[20 marks]

Question 2. Let $\Sigma'$ be the signature with one sort, one constant and two binary operations. Let $B$ be the $\Sigma'$-algebra of natural numbers with 0, addition and multiplication.

Show that there exist exactly two endomorphisms on $B$.

[25 marks]
Question 3. Produce an initial specification for the algebra of queues of natural numbers that contains (among others) a sort queue, a constant emptyqueue for the empty queue and the following operations:

- **add**: adds a number at the end of the queue;
- **front**: returns the front element of a nonempty queue (if the queue is empty the number 0 shall be returned);
- **back**: removes the front element from a nonempty queue (if the queue is empty the empty queue shall be returned);
- **length**: computes the length of a queue.

[25 marks]

Question 4. Let s be a sort, 1: s a constant, *, exp: nat × s → s operations (* used in infix notation), and x, y, z: s variables. Consider the term rewriting system R given by the following rules:

\[
\begin{array}{l}
x \times 1 \rightarrow x \\
1 \times x \rightarrow x \\
\exp(1, x) \rightarrow 1 \\
\exp(x, 1) \rightarrow x \\
\exp(x \times y, z) \rightarrow \exp(x, z) \times \exp(y, z) \\
\exp(x, y \times z) \rightarrow \exp(\exp(x, y), z)
\end{array}
\]

Prove that R is confluent and terminating. [30 marks]

Date due: Tuesday, 12 December 2006