Question 1. Let \( \Sigma \) be the signature consisting of one sort \( s \), a constant \( c : s \) and one binary operation \( f : s \times s \to s \). Consider the following \( \Sigma \)-algebras \( A, B, C \):

\[
A_s := \mathbb{R} = \text{the set of real numbers},
\]
\[
c^A := 0,
\]
\[
f^A(r_1, r_2) := r_1 + r_2 \ (r_1, r_2 \in \mathbb{R}), \text{ that is, } f^A \text{ is addition.}
\]

\[
B_s := \mathbb{R}_0^+ = \{ r \in \mathbb{R} \mid r \geq 0 \},
\]
\[
c^B := 0,
\]
\[
f^B(r_1, r_2) := r_1 + r_2 \ (r_1, r_2 \in \mathbb{R}_0^+).
\]

\[
C_s := \mathbb{R}^+ = \{ r \in \mathbb{R} \mid r > 0 \},
\]
\[
c^C := 1,
\]
\[
f^C(r_1, r_2) := r_1 \cdot r_2 \ (r_1, r_2 \in \mathbb{R}^+), \text{ that is, } f^C \text{ is multiplication.}
\]

For each of the algebras \( B \) and \( C \) decide whether or not it is isomorphic to the algebra \( A \). Justify your answers. \[20 \text{ marks}\]

Question 2.

(a) Produce an initial specification for the algebra of nonempty finite sets of natural numbers that has the following operations:

- \text{singleton}: creating a set with one element;
- \text{union}: the union of two sets;
- \text{minset}/\text{maxset}: the least/greatest element of a set;
- \text{member}: testing membership.
- \text{card}: the cardinality of a set;

You may import suitable data types of natural numbers and booleans, and add further operations. You may also use familiar mathematical notation. For example, you may write \( \{x\} \) for \text{singleton}(x), \( s \cup t \) for \text{union}(s, t), e.t.c. \[15 \text{ marks}\]

(b) Determine a minimal set of generators in the specification you have given in (a) above. Are these generators free? \[5 \text{ marks}\]
(c) Suppose you added an operation \texttt{count} : \texttt{set} → \texttt{nat} and the following equations to your initial specification:

\begin{align*}
\text{count}([x]) &= 1 \\
\text{count}(s \cup t) &= \text{count}(s) + \text{count}(t)
\end{align*}

Explain why the extended specification would be flawed. \hspace{1cm} \text{[5 marks]}

(d) Give a Haskell implementation, based on binary labelled trees, of the ADT you specified in (a) such that the operations \texttt{singleton}, \texttt{minset} and \texttt{maxset} run in constant time and \texttt{member} runs in logarithmic time, for sets represented as nearly balanced trees. Estimate the runtimes of your implementations of \texttt{union} and \texttt{card}. \hspace{1cm} \text{[25 marks]}

\textbf{Question 3.} Let \( s \) be a sort, \( 1 : s \) a constant, \( *, \exp : \texttt{nat} \times s \to s \) operations (* used in infix notation), and \( x, y, z : s \) variables. Consider the term rewriting system \( R \) given by the following rules:

\begin{align*}
\text{\( x \times 1 \mapsto x \)} \\
\text{\( 1 \times x \mapsto x \)} \\
\text{\( \exp(1, x) \mapsto 1 \)} \\
\text{\( \exp(x, 1) \mapsto x \)} \\
\text{\( \exp(x \times y, z) \mapsto \exp(x, z) \times \exp(y, z) \)} \\
\text{\( \exp(x, y \times z) \mapsto \exp(\exp(x, y), z) \)}
\end{align*}

Prove that \( R \) is confluent and terminating. \hspace{1cm} \text{[30 marks]}

\textbf{Date due: 13 December 2005}