Special morphisms in categories: Cancellation and Inversion

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Introduction

We consider

- monomorphisms
- epimorphisms
- coretractions (“split monomorphisms”)
- retractions (“split epimorphisms”)
- isomorphisms;

first in abstract categories, and then in our main generic examples (the three classes of small categories, the five central large categories, and functor categories).

ALL UNPROVEN STATEMENTS ARE EXERCISES.

Remarks are excluded, all other assertions have elementary easy proofs.

Overview

1. Special morphisms
2. Investigating the three types of small categories
3. Investigating the main large categories
4. Functors and natural transformations

Monomorphisms and epimorphisms

Let $\mathfrak{C}$ be a category, and $f : X \to Y$ a morphism in $\mathfrak{C}$:

- f is a **monomorphism** if for all $\alpha, \beta : Z \to X$ we have

$$f \circ \alpha = f \circ \beta \Rightarrow \alpha = \beta.$$

- f is an **epimorphism** if for all $\alpha, \beta : Y \to Z$ we have

$$\alpha \circ f = \beta \circ f \Rightarrow \alpha = \beta.$$

Thus "monomorphism" means **left-cancellable**, and "epimorphism" means **right-cancellable.** The notion of mono and epi are dual to each other:

- f is mono in $\mathfrak{C}$ iff f is epi in $\mathfrak{C}^{\mathrm{t}}$.
- f is epi in $\mathfrak{C}$ iff f is mono in $\mathfrak{C}^{\mathrm{t}}$.

Reflection of monos and epis

A functor $F : \mathfrak{C} \to \mathfrak{D}$ is **faithful** iff for all morphisms $f, g : X \to Y$ in $\mathfrak{C}$ holds:

$$F(f) = F(g) \Rightarrow f = g.$$

And F **reflects monomorphisms** resp. **reflects epimorphisms** if always from $F(f)$ mono resp. epi follows that also f is mono resp. epi.
(If we have the other direction, then we say that F **preserves mono-/epimorphisms**.)

Lemma *A faithful functor reflects monomorphisms and epimorphisms.*

(Remark: Leftadjoints preserve epimorphisms, rightadjoints preserve monomorphisms.)

Retractions, coretractions, and isomorphisms

Consider a category $\mathfrak{C}$ and morphisms $f : X \to Y$, $g : Y \to X$ in $\mathfrak{C}$:

- g is a **left-inverse** of f iff $g \circ f = \mathrm{id}_X$;
- g is a **right-inverse** of f iff $f \circ g = \mathrm{id}_Y$.

The notions of left- and right-inverse are dual to each other: g is a left-inverse of f in $\mathfrak{C}$ iff g is a right-inverse of f in $\mathfrak{C}^{\mathrm{t}}$, and vice-versa.

f is called a **retraction** if f has a right-inverse, and f is called a **coretraction** if f has a left-inverse. The notions of retraction and coretraction are dual to each other.

- g is an **inverse** of f iff g is left- and right-inverse of f.
- f is called an **isomorphism** iff f has an inverse.

Properties regarding inversion

We have:

1. The notion of isomorphism is self-dual.
2. Every morphism has at most one inverse.
3. A morphism is an isomorphism iff it is a retraction and a coretraction.

Regarding monos and epis:

- Every coretraction is a monomorphism.
- Every retraction is an epimorphism.

Lemma *Every functor preserves retractions, coretractions and isomorphisms.*

Free categories

In a free category cat(G) (where G is a dgg) we have:

1. Every morphism is monomorphism and epimorphism.
2. Every retraction and every coretraction is an identity.
3. Thus only the identities are isomorphisms.

Monoids

Consider a monoid $\mathcal{M} = (M, \circ, e)$ and $x \in M$:

1. x is a monomorphism resp. epimorphism in $\mathrm{cat}(\mathcal{M})$ iff x is left- resp. right cancellable in $\mathcal{M}$.
2. x is a coretraction resp. retraction in $\mathrm{cat}(\mathcal{M})$ iff x is left resp. right invertible in $\mathcal{M}$.
3. x is an isomorphism in $\mathrm{cat}(\mathcal{M})$ iff x is invertible in $\mathcal{M}$.

Since inverses are unique in arbitrary categories, we get that inverses are unique in monoids.

And from the preservation properties of functors we get, that homomorphisms preserve left- and right-invertibility as well as invertibility.

Quasi-ordered sets

Consider a qoset $\mathcal{M} = (M, \leq)$ and $(x, y) \in \leq$:

1. (x, y) is always monomorphism and epimorphism in $\mathrm{cat}(\mathcal{M})$.
2. The following conditions are equivalent:
   1. (x, y) is coretraction in $\mathrm{cat}(\mathcal{M})$.
   2. (x, y) is retraction in $\mathrm{cat}(\mathcal{M})$.
   3. (x, y) is isomorphism in $\mathrm{cat}(\mathcal{M})$.
   4. x and y are equivalent in $\mathcal{M}$, i.e., $x \leq y$ and $y \leq x$.

The category of sets

Consider a morphism $f : X \to Y$ in $\mathfrak{SET}$:

1. f is a monomorphism iff f is injective.
2. f is an epimorphism iff f is surjective.
3. f is a coretraction iff either $X = Y = \emptyset$, or $X \neq \emptyset$ and f is injective.
4. f is a retraction iff f is surjective.
5. f is an isomorphism iff f is bijective.

The last assertion is equivalent to the axiom of choice.

Application to concrete categories

Recall that for every concrete category $\mathfrak{C}$ we have the forgetful functor $V : \mathfrak{C} \to \mathfrak{SET}$.

Obviously V is faithful, and thus:

Lemma *For a concrete category all injective morphisms are monomorphisms, and all surjective morphisms are epimorphisms.*

The category of dgg's

Consider a morphism $f : G \to G'$ in $\mathfrak{DGG}$:

1. f is a monomorphism iff vertex map and edge map are injective.
2. f is an epimorphism iff vertex map and edge map are surjective.
3. Monomorphisms don't need to be coretractions (consider for example the embedding of the trivial dgg with one vertex into some dgg).
4. Epimorphisms don't need to be retractions (consider for example the epimorphism of some dgg to the dgg with one vertex and one loop).
5. Every bimorphism is an isomorphism, and thus $\mathfrak{DGG}$ is balanced.

The category of monoids

- The monomorphisms in $\mathfrak{MON}$ are exactly the injective homomorphisms.
- The isomorphisms in $\mathfrak{MON}$ are exactly the bijective homomorphisms.

There are epimorphisms in $\mathfrak{MON}$ which are not surjective: Consider the canonical embedding

$$j_{\mathbb{Z}} : (\mathbb{Z}, \cdot, 1) \to (\mathbb{Q}, \cdot, 1).$$

Now $j_{\mathbb{Z}}$ is an epimorphism, since a homomorphism $f : (\mathbb{Q}, \cdot, 1) \to (M, \circ, e)$ is determined by its images on integers, because for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$ we have

$$f(\frac{p}{q}) = f(p) \circ f(q)^{-1}$$

(using that homomorphisms preserve inverses and inverses are uniquely determined).

Bimorphisms

There are exactly three homomorphisms $f_1, f_2, f_3 : (\mathbb{Q}, \cdot, 1) \to (\mathbb{Z}, \cdot, 1)$:

1. $f_1(x) := 1$
2. $f_2(x) := f_1(x)$ except of $f_2(0) = 0$
3. $f_3(x) := f_2(x)$ except of $f_3(x) := -1$ if $x < 0$.

Thus $j_{\mathbb{Z}}$ is neither a retraction nor a coretraction.

A morphism f in a category $\mathfrak{C}$ is called a **bimorphism** if f is monomorphism and epimorphism. $\mathfrak{C}$ is **balanced** iff every bimorphism is an isomorphism.

So $\mathfrak{SET}$ is balanced, while $\mathfrak{MON}$ is not.

The category of quasi-ordered sets

- The monomorphisms in $\mathfrak{QORD}$ are exactly the injective homomorphisms.
- The epimorphisms in $\mathfrak{QORD}$ are exactly the surjective homomorphisms.
- The canonical embedding

$$j_2 : (\{1,2\}, \mathrm{id}_{\{1,2\}}) \to (\{1,2\}, \mathrm{id}_{\{1,2\}} \cup \{(1,2)\})$$

is a bimorphism which is not an isomorphism. Thus $\mathfrak{QORD}$ is not balanced.

The category of categories

Consider a morphism $F : \mathfrak{C} \to \mathfrak{D}$ in $\mathfrak{KAT}$:

- F is a monomorphism iff the object map and the morphism map of F are injective.
- If F is an epimorphism, then the object map of F is surjective.
- F is an isomorphism iff both object map and morphism map are bijective.

The monoid-homomorphism $j_{\mathbb{Z}} : (\mathbb{Z}, \cdot, 1) \to (\mathbb{Q}, \cdot, 1)$ yields a functor $\mathrm{cat}(j) : \mathrm{cat}((\mathbb{Z}, \cdot, 1)) \to \mathrm{cat}((\mathbb{Q}, \cdot, 1))$; this functor is a bimorphism, and its morphism map is not surjective.

Thus $\mathfrak{KAT}$ is not balanced.

(Note that the order-homomorphism j_2 (which is a bimorphism) yields a functor $\mathrm{cat}(j_2)$ which is monomorph but not epimorph.)

Embeddings

A functor for which both morphism map and object map are injective, is called an **embedding**. The following conditions are equivalent for a functor $F : \mathfrak{C} \to \mathfrak{D}$:

- F is an embedding.
- F is a monomorphism in $\mathfrak{KAT}$.
- The morphism map of F is injective.
- The object map of F is injective, and F is faithful.

Functor categories

Consider functors $F, G : \mathfrak{C} \to \mathfrak{D}$. It is said that a natural transformation $\eta : F \to G$ is **componentwise mono-/epimorph** (or also "pointwise") if for all $X \in \mathrm{Obj}(\mathfrak{C})$ the morphism $\eta_X : F(X) \to G(X)$ is monomorph resp. epimorph.

If the natural transformation $\eta : F \to G$ is componentwise mono-/epimorph, then η is monomorph resp. epimorph in $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$.

End