SAT and the Polya Permanent Problem

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SAT: Connecting combinatorics and linear algebra

Main results
Complement invariance
Lean clause-sets
Minimal unsatisfiability
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Hypergraph 2-colouring
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Overview

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A few general notations

- $c(F)$ for the number of clauses
- $n(F)$ for the number of variables
- $\top$ is the empty clause-set
- $\bot$ is the empty clause

Complement-invariant clause-sets

For clauses C and clause-sets F:

$$\overline{C} := \{\overline{x} : x \in C\}$$
$$\overline{F} := \{\overline{C} : C \in F\}.$$

F is called **complement-invariant** if $F = \overline{F}$.

F is complement-invariant iff
there is a clause-set F_0 with $F = F_0 \cup \overline{F_0}$.

Such a F_0 with $c(F_0) = \frac{c(F)}{2}$ is called a **core half**.

For example

$$F = \{\, \{a, \overline{b}\}, \{b, c\}, \{\overline{a}, b\}, \{\overline{b}, \overline{c}\} \,\}$$

is complement-invariant, with core halfs $\{\, \{a, \overline{b}\}, \{b, c\} \,\}$ or $\{\, \{a, \overline{b}\}, \{\overline{b}, \overline{c}\} \,\}$ for example.

The reduced deficiency

For a clause-set F the **deficiency** is

$$\boldsymbol{\delta(F)} := c(F) - n(F).$$

The **reduced deficiency** is

$$\boldsymbol{\delta_{\mathrm{r}}(F)} := \frac{1}{2}(\delta(F) - n(F)).$$

For a core half F_0 of F we have

$$\delta_{\mathrm{r}}(F) = \delta(F_0).$$

For the example $F = \{ \{a, \overline{b}\}, \{b, c\}, \{\overline{a}, b\}, \{\overline{b}, \overline{c}\} \}$ we have

$$\begin{aligned} \delta(F) &= 4 - 3 = 1 \\ \delta_{\mathrm{r}}(F) &= \frac{1}{2}(1 - 3) = -1 = (2 - 3). \end{aligned}$$

Autarkies and lean clause-sets

Recall:

- An **autarky** for a clause-set F is a partial assignment which satisfies every clause it touches.
- A clause-set is **lean** iff it has no non-trivial autarkies.
- The only satisfiable lean clause-set is $\top$.
- Every minimally unsatisfiable clause-set is lean.
- Other examples of lean clause-sets are obtain by Tseitin-extensions of lean clause-sets.
- The **lean kernel** of a clause-set F is the largest (subsumption-wise) lean sub-clause-set of F. It can also be obtained by repeated autarky reduction (actually also in one step).

The lean kernel of a clause-set consists of exactly all the clauses which can be used in some resolution refutation.

Computing the lean kernel in general

In general:

1. Deciding whether a clause-set is lean is coNP-complete.
2. The best algorithm for computing the lean kernel (thus also deciding leanness) seems to be given by
   - using a SAT solver which returns the set of variables used in a resolution refutation it found
   - this can be computed efficiently without much overhead; for example the `OKsolver` always does it (for intelligent backtracking); see [Kullmann, Silva, Lynce; SAT06].

Applications of the computation of the lean kernel:

1. for computing a MU sub-clause-set (which must be contained in the lean kernel);
2. for computing a MAXSAT sub-clause-set (which must be contained in the complement of the lean kernel).

Lean complement-invariant clause-sets

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Deciding leanness is coNP-complete also for complement-invariant clause-sets. However we have:

- If the complement-invariant F is lean, then $\delta_{\mathrm{r}}(F) \geq 0$.
- This is similar to the general situation, where we have $\delta(F) \geq 1$ for lean clause-sets.
- Actually, in the general situation, **matching leanness** suffices to establish $\delta(F) \geq 1$, while for complement-invariant **linear leanness** suffices.

So we can ask for deciding leanness of complement-invariant clause-sets F with $\delta_{\mathrm{r}}(F) = 0$. Now, transferring [Robertson, Seymour, Thomas 1999; McCuaig 2004], we get

Theorem *Deciding whether a complement-invariant F with $\delta_{\mathrm{r}}(F) = 0$ is lean is decidable in polynomial time.*

Generalising minimal unsatisfiability

Considering an arbitrary **autarky systems** (which allow for specialisations of autarkies) we have the following notions for clause-sets F:

1. F is **$\mathcal{A}$-satisfiable** iff the $\mathcal{A}$-lean kernel of F is $\top$.
2. F is **minimally $\mathcal{A}$-unsatisfiable** iff F is $\mathcal{A}$-unsatisfiable, while every strict sub-clause-set of F is $\mathcal{A}$-satisfiable.
3. F is **barely $\mathcal{A}$-lean** iff A is $\mathcal{A}$-lean, but after removal of any clause this is no longer the case.
4. F is **$\mathcal{A}$-autarky indecomposable** iff F cannot be obtained by a variable-disjoint union of two $\mathcal{A}$-lean clause-sets F_1, F_2, where to the clauses of F_2 literals x with $\mathrm{var}(x) \in \mathrm{var}(F_1)$ can be added (arbitrarily).

For the full autarky system, $\mathcal{A}$-satisfiability is satisfiability, and minimal $\mathcal{A}$-unsatisfiability is minimal unsatisfiability.

Characterising minimal $\mathcal{A}$-unsatisfiability

Trivially, every minimally unsatisfiable clause-set is lean. We can get an equivalence, and this for arbitrary autarky systems $\mathcal{A}$, as follows:

Lemma *Consider a normal autarky system $\mathcal{A}$. Then a clause-set F is minimally $\mathcal{A}$-unsatisfiable if and only if*

1. *F is barely $\mathcal{A}$-lean, and*
2. *F is $\mathcal{A}$-indecomposable.*

This lemma might be useful even for the full autarky system. We will apply it to the autarky system given by linear autarkies.

Deciding minimal unsatisfiability

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Deciding whether a general F or a complement-invariant F is minimally unsatisfiable is D^P-complete.

Theorem *Deciding whether a complement-invariant clause-set F with $\delta_r(F) = 0$ is minimally unsatisfiable can be done in polynomial time.*

Remark: While in general performing reduction by linear autarkies needs the power of linear programming, here actually we only need to consider **balanced linear autarkies**, which can be handled more efficiently by just considering systems of linear *equations*.

SAT for complement-invariant clause-sets

Finally, what about SAT decision for (arbitrary) complement-invariant clause-sets?

- Obviously, still SAT decision is NP-complete.
- Specialisation to $\delta_r(F) = 0$ doesn't help (padding).

Recall:

1. The same problem arises for arbitrary clause-sets F and the (normal) deficiency.
2. Cure: the **maximal deficiency** $\delta^*(F)$.

Analogously, we introduce the **maximal reduced deficiency**

$$\boldsymbol{\delta_r^*(F)} := \max_{F' \subseteq F} \delta_r(F).$$

For a core half F_0 of F we have

$$\delta_r^*(F) = \delta^*(F_0).$$

Polynomial time SAT decision

Constructivising [Robertson, Seymour, Thomas; McCuaig] we should be able to prove:

Conjecture The following functional computation problem can be solved in polynomial time: Given a square matrix A over $\{-1, 0, +1\}$, if A is not an SNS-matrix, then a singular matrix $A' \in \mathfrak{Q}(A)$ over $\mathbb{Z}$ can be computed.

Theorem *Assume that the conjecture holds true. Then for complement-invariant clause-sets F with $\delta_r^*(F) = 0$ the satisfiability problem is decidable in polynomial time (providing also a satisfying assignment).*

It follows that if F is unsatisfiable, then a minimally unsatisfiable sub-clause-set can be computed in polynomial time. (*Open*: Can we also compute a *minimum* minimally unsatisfiable sub-clause-set?!?)

The main conjecture

Conjecture *For fixed $k \in \mathbb{N}_0$, the satisfiability problem for complementation-invariant clause-sets F with $\delta_r^*(F) \le k$ is decidable in polynomial time.*

The underlying combinatorial structures is of good interest to the graph theory community and the combinatorics community; results in this direction could have two bearings:

- embedding the (narrow) graph theoretical / combinatorial problems into the (richer) satisfiability problem makes certain operations much more transparent — hopefully a better understanding also for the original problems is finally reached;
- embedding the (rich) graph theoretical / combinatorial problem into the (slim) satisfiability problem adds combinatorial structure to SAT.

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Hypergraph 2-colouring

- A (finite) hypergraph is a pair $G = (V, E)$, where V is the (finite) vertex set, and E is a set of subsets of V.
- A 2-colouring of G is a map $f : V \to \{0, 1\}$ such that no hyperedge is monochromatic, i.e., for every $H \in E$ there are $v, w \in H$ with $f(v) = 0$ and $f(w) = 1$.

Translating the hypergraph 2-colouring problem into a SAT problem is done trivially by creating a "positive copy" and a "negative copy" of the hyperedges, for example the hypergraph (given by its hyperedges)

$$\{ \{a, b\}, \{b, c\} \}$$

is translated into the clause-set

$$\{ \{v_a, v_b\}, \{v_b, v_c\}, \ \{\overline{v_a}, \overline{v_b}\}, \{\overline{v_b}, \overline{v_c}\} \}.$$

Hypergraphs vs. complement-invariance

So the hypergraph 2-colouring problem is **exactly** the same as the SAT-problem for complement-invariant clause-sets which are also **PN clause-sets**, that is, every clause is either positive or negative.

How much more general is SAT for (general) complement-invariant clause-sets compared to PN complement-invariant clause-sets?

1. The standard translation of arbitrary clause-sets into PN clause-sets (by introducing new variables) can be symmetrised, and then yields a translation of complement-invariant clause-sets into PN complement-invariant clause-sets.
2. This translation is not only SAT-preserving, but also MU-, autarky- and $\#$SAT-preserving.

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Why not just hypergraphs?

So, modulo a very simple (AC^0) and very well behaved reduction, complement-invariant clause-sets and PN complement-invariant clause-sets can be identified w.r.t. SAT and variations.
However, there is one central asset of SAT, namely resolution:

- Resolution, **symmetrised**, does not leave the class of complement-invariant clause-sets.
- However, it leaves the class of PN complement-invariant clause-sets!

Using special satisfying assignments

Consider some restricted form of satisfying assignments, via a predicate $S(\varphi, F)$, which is true if the partial assignment φ satisfies F and fulfils the special requirement, e.g.,

- φ is matching satisfying (recall Stefan Szeider's talk)
- φ is NAESAT-satisfying, i.e., in every clause there is not only a satisfied literal, but also a falsified one.

You obtain an **autarky system**: A "S-autarky" for F is a partial assignment φ such that φ satisfies $F[\mathrm{var}(\varphi)]$, where for a set V of variables $F[V]$ denotes the restriction of F to V.

- From matching satisfying assignments we get **matching autarkies.**
- From NAESAT-satisfying assignments we get **balanced autarkies.**

Balanced autarkies

Consider a clause-set F and its "symmetrised" version $F := F_0 \cup \overline{F_0}$:

1. F_0 is satisfiable w.r.t. balanced autarkies, i.e. NAESAT-satisfiable, iff F is satisfiable.
2. F_0 is lean w.r.t. balanced autarkies iff F is lean.
3. F_0 is minimally unsatisfiable w.r.t. balanced autarkies iff F is minimally unsatisfiable.

Thus the symmetrisation operation translates between

- the general property for complement-invariant $F_0 \cup \overline{F_0}$
- the balanced version of the property for general F_0.

In this way we can make use of "both worlds": The "balanced SAT-properties" of arbitrary clause-sets vs. the "general SAT-properties" of complement-invariant clause-sets.

Using the clause-variable matrix

For a clause-set F let $M(F)$ denote the **clause-variable matrix**:

1. As was first noticed by [Davydov/Dovydova], that F is satisfiable can be expressed in a natural way by $M(F)$ using "matrix speech".
2. Various versions can be given, exploiting duality from the realm of linear programming.

All these properties are best expressed using the notions of **qualitative matrix analysis**:

- That a variable-clause matrix is balanced lean is exactly the same as being an L-matrix,
- where the simplest case, where the matrix is square (deficiency = 0) is called **SNS-matrix.**

The origins of qualitative matrix analysis

"Qualitative matrix analysis" (QMA) is matrix analysis modulo the equivalence relation between matrices given by having the same sign pattern.

[Brualdi, Shader: Matrices of sign-solvable linear system (1995)] describes the foundations of QMA in qualitative economics:

> *Qualitative economics is usually considered to have originated with the work of Samuelson who discussed the possibility of determining unambiguously the qualitative behavior of solution values of a system of equations. In his pioneering paper Lancaster put it this way:*

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Origins continued

Economists believed for a very long time, and most economists would still hope it to be so, that a considerable body of sensible economic propositions could be expressed in a qualitative way, that is, in a form in which the algebraic sign of some effect is predicted from a knowledge of the signs, only, of the relevant structural parameters of the system.

We consider here the fundamental notion, that a square real matrix A has a **sign-invariant determinant**, that is, for all matrices A' with the same sign pattern as A we have $\operatorname{sgn}(\det(A')) = \operatorname{sgn}(\det(A))$.

Sign patterns and zero patterns

- The *sign pattern* $\mathrm{sgn}(M)$ of matrix M is the $\{-1, 0, +1\}$-matrix $\mathrm{sgn}(M)$ of the same dimension given by entrywise sgn-formation.
- The **qualitative class** of M, $\boldsymbol{\mathfrak{Q}(M)} := \{M' : \mathrm{sgn}(M') = \mathrm{sgn}(M)\}$, is the set of all matrices with the same sign pattern.
- The *null pattern* of M is $\mathrm{sgn}(|M|)$ (a $\{0, 1\}$-matrix), where $|M|$ denotes entrywise absolute-value formation.

For example:

$$\begin{aligned} M &= \begin{pmatrix} 4 & -5 \\ 0 & 2 \end{pmatrix} \\ \mathrm{sgn}(M) &= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \\ \mathrm{sgn}(|M|) &= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

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SNS-matrices

By definition, A is has sign-invariant determinant if and only if

(i) either $\forall\, A' \in Q(A) : \det(A') = 0$

(ii) or $\forall\, A' \in Q(A) : \det(A') > 0$

(iii) or $\forall\, A' \in Q(A) : \det(A') < 0$.

For every matrix $\mathfrak{Q}(A)$ is connected, while $\det : \mathfrak{Q}(A) \to \mathbb{R}$ is continuous, and so $\det(\mathfrak{Q}(A))$ is connected.

Thus cases (ii) or (iii) hold iff $\forall\, A' \in Q(A) : \det(A') \neq 0$, in which case A is called an **SNS-matrix** ("sign-non-singular").

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Diagonals

For a square matrix A of order n:

- The **diagonal** corresponding to permutation $\pi \in S_n$ is the vector $(A_{i,\pi(i)})_{i \in \{1,\dots,n\}}$.
- The **main diagonal** is the diagonal corresponding to the identity map.
- A diagonal is **non-null** if *all* entries are non-zero.

For example, the permutation $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \in S_3$

corresponds to the diagonal

$$\begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}.$$

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The Leibniz determinant expansion

For a square matrix A of order n we have

$$\det(A) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \cdot \prod_{i=1}^{n} A_{i,\pi(i)}.$$

- In other words, we run through all diagonals of A and sum up their products, weighted by the sign of the corresponding permutation.
- Clearly we need to consider only non-null diagonals (they correspond to the non-null terms in the determinant expansion).

Remark: The determinant expansion does not yield a poly-time computation of the determinant, this however can be achieved by Gaussian elimination.

The determinant expansion for order 3

The three even diagonals of a square matrix of order 3 are

$$\begin{pmatrix} \bullet & & \\ & \bullet & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & \bullet & \\ & & \bullet \\ \bullet & & \end{pmatrix}, \begin{pmatrix} & & \bullet \\ \bullet & & \\ & \bullet & \end{pmatrix},$$

and the three odd diagonals are

$$\begin{pmatrix} & \bullet & \\ \bullet & & \\ & & \bullet \end{pmatrix}, \begin{pmatrix} & & \bullet \\ & \bullet & \\ \bullet & & \end{pmatrix}, \begin{pmatrix} \bullet & & \\ & & \bullet \\ & \bullet & \end{pmatrix}.$$

(Only the non-null diagonals need to be considered for a given matrix, that is, where all entries are not zero.)

First characterisation

Lemma *For a square matrix A we have*

(i) *A has sign-invariant determinant with sign* $\varepsilon \in \{\pm 1\}$ *if and only if every non-null term in the determinant expansion has sign* ε*, and there is one such term.*

(ii) *A has sign-invariant determinant with sign* 0 *iff there is no non-null term in the determinant expansion.*

Proof: For Part (i) use Laplace determinant expansion after a row, where two terms with positive resp. negative sign differ, to see that in $\mathfrak{Q}(A)$ a positive and a negative sign of the determinant can be realised.

Do the same also for Part (ii), using that the existence of a non-null term implies the existence of a positive as well as a negative term (if the determinant is zero). $\surd$

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An example

Let's consider $n = 3$. The even resp. odd permutations in S_3 are

$$123, 231, 312$$
$$321, 132, 213.$$

Now

$$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 0 \\ -1 & -1 & -1 \end{pmatrix}$$

is an SNS-matrices (all four non-null term are -1), while

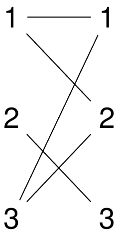
$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

is not.

Associated bipartite graphs

For a $\{0,1\}$-matrix A of dimension $n \times m$ let $\mathbf{bip}(\boldsymbol{A})$ be the bipartite graph with bipartition $(\{1,\dots,n\},\{1,\dots,m\})$ and edges $\{i,j\}$ for $A_{i,j} = 1$.

So the non-null diagonals of a square matrix A correspond 1-1 to the *perfect matchings* of $\mathrm{bip}(\mathrm{sgn}(|A|))$, for example $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ yields



with two perfect matchings

$$\{\,\{1,1\},\{2,3\},\{3,2\}\,\},$$
$$\{\,\{1,2\},\{2,3\},\{3,1\}\,\}.$$

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Sign-singular matrices

- By the Lemma we get that a matrix A is sign-singular if and only if bip(A) does not have a perfect matching.
- This can be checked in polynomial time. It remains the (much) harder task of deciding whether a matrix A is SNS.

BTW, the generalisation of property SNS to matrices of arbitrary dimension ("L-matrices") is coNP-complete.

The Permanent

For a square matrix A of order n we define

$$\mathrm{per}(A) = \sum_{\pi \in S_n} \prod_{i=1}^{n} A_{i,\pi(i)}.$$

- So the permanent is defined by an expansion like the determinant, only that we do not take the sign of the permutations into account.
- The permanent of a square $\{0,1\}$-matrix A is the number of perfect matchings of $\mathrm{bip}(A)$ (a #P-complete problem).

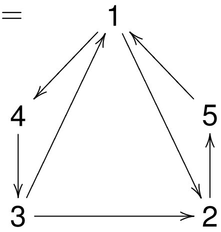
By the previous lemma we get:

A square matrix A is SNS if and only if $\det(A) \neq 0$ and $\mathrm{per}(|A|) = |\det(A)|$.

$\{0,1\}$-matrices as adjacency-matrices

For a square $\{0,1\}$-matrix A of order n let **dgl(A)** denote the directed graph with vertex set $\{1,\dots,n\}$ and an edge (i,j) if $A_{i,j} = 1$, while by **dg(A)** we denote the irreflexive version of dgl(A) (ignoring the entries of A on the main diagonal). For example

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \mathrm{dg}(A) =$$



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Even cycles in digraphs

Every square matrix A with $\det(A) \neq 0$ must have a non-zero diagonal. So by column permutation, w.l.o.g. for SNS-decision we can assume a non-zero main diagonal.

Now for a square $\{0, 1\}$-matrix A with non-zero diagonal the following assertions are equivalent:

1. A is SNS.
2. $\text{per}(A) = \det(A)$.
3. Each term in the determinant expansion is ≥ 0.
4. There is no odd non-zero diagonal.
5. $\text{dgl}(A)$ ($\text{dg}(A)$) has no even cycle.

Proof: A cyclic permutation $(a_1, \ldots, a_n)$ is odd iff n is even; n is the length of the corresponding cycle in $\text{dgl}(A)$.

A general permutation is odd iff the cycle decomposition has an odd number of odd cyclic permutations.

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Counter-examples as autarkies

For a square matrix A:

A is SNS iff
$\forall A' \in \mathfrak{Q}(A) \, \neg\exists \, x \neq 0 : A'x = 0$. Thus

A is not SNS iff
$\exists \, A' \in \mathfrak{Q}(A) \, \exists \, x \neq 0 : A'x = 0$.

W.l.o.g.: $A = \mathrm{sgn}(A)$, i.e., A is $\{-1, 0, +1\}$-matrix. What *is* a $\{-1, 0, +1\}$-matrix ?!

A **(multi-)clause-set** !

A is the clause-variable matrix of a clause-set F. x is interpreted as a partial assignment (zero entries mean unassigned), and yields an **autarky** — every clause which is touched is satisfied.

Moreover, in every touched clause there must be also a falsified literal — so its a **balanced autarky**.

Example

Consider the non-SNS matrix $\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. The clause-set F_0 with this clause-variable matrix is

$$F_0 := \{ \{a, \overline{b}\}, \{b, c\}, \{a, c\} \},$$

which has exactly two balanced autarkies, namely $\varphi_1 := \langle c \to 1, a, b \to 0 \rangle$ and $\varphi_2 := \langle c \to 0, a, b \to 1 \rangle$. The corresponding solution to $M(F) \cdot x = 0$ is

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = 0.$$

(Remark: Here we can use $M(F_0)$ itself, and thus φ_1, φ_2 are balanced *linear* autarkies.)

On the importance of SNS-matrix decision

As we have seen now, the following decision problems are all equivalent to the SNS-matrix decision problem by simple reductions:

1. Is a complement-invariant clause-set F with $\delta_r(F) = 0$ lean?
2. Is a clause-set F with $\delta(F) = 0$ balanced lean?
3. Is a PN clause-set F with $\delta(F) = 0$ balanced lean?
4. Given a directed graph, does it has no directed circuits of even length?

We have also seen that leanness is closely related to being minimally unsatisfiable (here), which for hypergraphs translates into the property of being minimally non-2-colourable.

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The original Pólya problem

Expanding the arguments we have given, one can also show the close relation to the following problem posed by Pólya (1913):

Given a $\{0, 1\}$-square matrix *A*, when can some of the 1's be changed to -1's in such a way that the permanent of *A* equals the determinant of the modified matrix?

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The solution (of many problems)
Conclusions and Outlook

The solution

- In 1996 independently by Robertson, Seymour, Thomas and McCuaig a polynomial time algorithm for the Pólya problem was demonstrated.
- Thus also SNS-matrix decision can be done in polynomial time.
- (Since the argumentation is quite complicated (40 pages of dense argumentation), it is not really clear whether, in case the matrix was found not to be an SNS-matrix, also a witness for this can be computed efficiently.)

Main results
Complement invariance
Lean clause-sets
Minimal unsatisfiability
SAT decision
Hypergraph 2-colouring
Autarkies
Constructing autarky systems
L-matrices and SNS-matrices
Qualitative matrix analysis
Origins
SNS-matrices
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Summary

- The class of **complementation-invariant clause-sets** has been introduced, together with the notion of **reduced deficiency**.
- Poly-time time SAT decision was shown (relative to a believable conjecture) for complementation-invariant clause-sets of maximal **reduced deficiency zero**.
- Many relations to graph theory, hypergraph theory, combinatorics, matrix analysis — the link provided by **autarky theory**.
- The main **conjecture** should provide a fertile ground for interesting investigations into SAT algorithms, which should find the attention of the wider combinatorics community.

End