Understanding minimal unsatisfiability (Extended Abstract on some recent developments)

Oliver Kullmann

Computer Science Department
Swansea University

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MU

- A literal is a variable or a negated variable.
- A clause is a finite set of literals, as their disjunction; we disallow complementary literals here.
- A clause-set is a set of clauses, as their conjunction; only finite clause-sets considered here.
- $\mathcal{MU}$ is the set of clause-sets, which are unsatisfiable, while removal of any clause renders them satisfiable.
- $n(F)$ is the number of (occurring) variables.
- $c(F)$ is the number of clauses.
- $\delta(F) := c(F) - n(F) \in \mathbb{Z}$ is the **deficiency**.

See Handbook Chapter Kleine Büning and Kullmann [8].

“Tarsi’s Lemma”

For example

$$\{\, \{\overline{a}\}, \{a, \overline{b}\}, \{a, b, \overline{c}\}, \{a, b, c\} \,\} \in \mathcal{MU}$$

with $\delta(F) = 4 - 3 = 1$.

$$\forall F \in \mathcal{MU} : \delta(F) \geq 1.$$

- Best known proof Aharoni and Linial [1].
- For an overview see the introduction of [12].

Deficiency for MU yields a complexity parameter.

- MU decision poly-time for fixed k (Fleischner, Kullmann, and Szeider [3]).
- Indeed fpt (Szeider [13]).

Degrees

The most basic information about MU is given by some knowledge on the *degrees*.

Literal degrees:

$$\mathrm{ld}_F(x) := |\{C \in F : x \in C\}|$$

Variable degrees:

$$\mathrm{vd}_F(v) := \mathrm{ld}_F(v) + \mathrm{ld}_F(\overline{v}).$$

MU(1) I

$$\mathcal{MU}_{\delta=1} = \{F \in \mathcal{MU} : \delta(F) = 1\}$$

- These are nice formulas, with a surprising number of applications.
- In the SAT world, classification due to Aharoni and Linial [1], Davydov, Davydova, and Kleine Büning [2].
- Indeed, independently equivalent classifications in different areas have been obtained; see [12].

Complete generation process:

1. Start with $F = \{\bot\} \in \mathcal{MU}_{\delta=1}$.
2. For $F \in \mathcal{MU}_{\delta=1}$:
    1. choose $C \in F$ and a variable $v \notin \mathrm{var}(F)$,
    2. choose D_1, D_2 with $D_1 \cup D_2 = C$,
    3. replace C by $D_1 \cup \{v\}, D_2 \cup \{\overline{v}\}$.

MU(1) II

Characterisation becomes MUCH easier, once you know that for all $F \in \mathcal{MU}_{\delta=1}$, $n(F) \neq 0$, there exists $v \in \mathrm{var}(F)$ with $\mathrm{vd}_F(v) \leq 2$.

We express this as

$$\mathrm{VDM}(1) = 2.$$

Of course, there are relevant open problems already here!

- For example concerning the *uniform* elements of $\mathcal{MU}_{\delta=1}$; Hoory and Szeider [6], Gebauer, Szabo, and Tardos [4].
- Uniformity (constant clause-length) features a lot in hypergraph theory, while we work mostly in the unrestricted setting.

MU(2) I

$$\mathcal{MU}_{\delta=2} = \{F \in \mathcal{MU} : \delta(F) = 2\}$$

- The basic characterisation is due to Kleine Büning [7].
- This concerns *nonsingular* elements of $\mathcal{MU}_{\delta=2}$ — every variable occurs at least twice positively as well as negatively.

$$\Big\{ \{v_1 \to v_2\}, \dots, \{v_{n-1} \to v_n\}, \{v_n \to v_1\}$$
$$\{v_1, \dots, v_n\}, \{\overline{v_1}, \dots, \overline{v_n}\} \Big\}$$

MU(2) II

The main open question here is:

Extend the generalisation to all of $\mathcal{MU}_{\delta=2}$ —
in a sense fusing the characterisations obtained for $\delta = 1, 2$.

This is needed for a better understanding of higher deficiencies.

Again, a fundamental step is to to show
$\forall\, F \in \mathcal{MU}_{\delta=2}, n(F) \neq 0 \;\exists v \in \mathrm{var}(F) : \mathrm{vd}_F(v) \leq 4.$

I.e., $\mathrm{VDM}(2) = 4$.

Finite Patterns Conjecture

For every k,
$\mathcal{MU}_{\delta=k}$ can be characterised by
***finitely many patterns*.**

The next frontier is $\mathcal{MU}_{\delta=3}$.

Min-var-degree

In general (F a clause-set, $\mathcal{C}$ a class of clause-sets):

$$\begin{aligned} \mu\mathrm{vd}(F) &:= \min_{v \in \mathrm{var}(F)} \mathrm{vd}_F(v) \\ \mu\mathrm{vd}(\mathcal{C}) &:= \max_{F \in \mathcal{C}} \mu\mathrm{vd}(F) \\ \mathrm{VDM}(k) &:= \mu\mathrm{vd}(\mathcal{MU}_{\delta=k}). \end{aligned}$$

In [9] the fundamental bound

$$\forall\, k \geq 1 : \mathrm{VDM}(k) \leq 2k$$

was shown.

Improving the bound

In [10] the upper bound $\mathrm{VDM}(k) \le 2k$ was improved to

$$\mathrm{VDM}(k) \le 1 + k + \log_2(k).$$

- Indeed a precise number-theoretical function $\mathrm{nM}(k)$ yields the upper bound.
- This upper bound is not sharp, and the first deficiency needing a correction is $k = 6$.

The main open problem here is the precise determination of $\mathrm{VDM}(k)$.

The sharpenings we produced unearth interesting aspects of MU; see [12] for further information.

Full clauses

An interesting combinatorial quantity for a clause-set F is the *number of full clauses*:

$$\mathrm{fc}(F) := |\{C \in F : \mathrm{var}(C) = \mathrm{var}(F)\}| \in \mathbb{N}_0.$$

We have

$$\mathrm{fc}(F) \leq \mu\mathrm{vd}(F).$$

So maximising the number of full clauses yields *lower bounds* on $\mathrm{VDM}(k)$.

Let $\mathrm{FCM}(k)$ be the maximum of $\mathrm{fc}(F)$ for $F \in \mathcal{MU}_{\delta=k}$.

Thus $\mathrm{FCM} \leq \mathrm{VDM}$.

Hitting clause-sets

Indeed it helps a lot to consider *hitting clause-sets* here:

Every two clauses have a clash.

If a hitting clause-set is unsatisfiable, it is automatically MU.

$\mathrm{VDH}(k), \mathrm{FCH}(k)$ denote the
maximal min-var-degree resp. number of full clauses
for hitting MU.

- We conjecture $\mathrm{VDH} = \mathrm{VDM}$.
- But definitely only $\mathrm{FCH} \leq \mathrm{FCM}$.

Meta-Fibonacci

We show

$$S_2 \leq \mathrm{FCH}$$

for the number-theoretic function S_2 ([11]).

And indeed we conjecture equality.

- Interesting recursion-theoretic phenomena show up (at the microscopic level).
- Belong to the field of “meta-Fibonacci” functions (nested recursive calls), as introduced by Hofstadter [5].

The four fundamental quantities

To summarise:

The quantities $\mathrm{VDM}(k), \mathrm{FCM}(k), \mathrm{VDH}(k), \mathrm{FCH}(k)$
seem interesting beasts: offering a lot of depth
— and good attack points!

The *precise* quantities matter here, and relevant number-theoretical functions appear.

Summary and outlook

I The main conjecture is the Finite Patterns Conjecture. Once we settled $\delta = 3$, we hopefully see better.

II Studying the "four fundamental quantities" reveals surprising structures — if you go for the exact determination.

Outlook (no time to discuss in this talk):

I **Reductions** are an important tool for understanding MU (for example eliminating singular variables).

II First you concentrate on understanding (only) reduced cases, and then you extend.

III The reductions have good properties, and likely there is much more to come.

End

(references on the remaining slides).

For my papers see
http://cs.swan.ac.uk/~csoliver/papers.html.

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