On Davis-Putnam reductions for minimally unsatisfiable clause-sets

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Two sources: DP and MU

- Under what circumstances is DP-reduction (single steps of the DP-reduction from Davis and Putnam [2]; also called "variable elimination") confluent or confluent modulo isomorphism?
- Further steps towards the classification of all minimally unsatisfiable clause-sets.

Bibliographical remarks

- Underlying paper is Kullmann and Zhao [9].
- The (extended) underlying technical report is Kullmann and Zhao [10] (containing also some technical corrections).

Outline

Clause-sets

- Literals are variables v and their complements $\overline{\mathbf{v}}$.
- A clause C is a finite and clash-free set of literals, i.e., $C \cap \overline{C} = \emptyset$.
- A **clause-set** is a finite set of clauses.
- The set of all clause-sets is $\mathcal{CLS}$.

For example

$$\mathcal{F}_2 = \{ \{v_1, v_2\}, \{\overline{v_1}, \overline{v_2}\}, \{\overline{v_1}, v_2\}, \{\overline{v_2}, v_1\} \}$$

is a clause-set (minimally unsatisfiable, deficiency 2).

Remark: Clause-sets are considered as (precise) combinatorial objects, as generalised hypergraphs.

Applying partial assignments

The application of a partial assignment $\varphi \in \mathcal{PASS}$ to a clause-set $F \in \mathcal{CLS}$ is denoted by

$$\boldsymbol{\varphi * F} \in \mathcal{CLS}.$$

Satisfied clauses are removed, then falsified literals.
A special clause-set is $\top := \emptyset$, a special clause is $\bot := \emptyset$.

- F is **satisfiable** iff there is $\varphi \in \mathcal{PASS}$ with $\varphi * F = \top$.
- $\top$ is satisfiable.
- $\{\bot\}$ is unsatisfiable.
- More generally, every F with $\bot \in F$ is unsatisfiable.

$$\mathcal{CLS} = \mathcal{SAT} \mathbin{\dot\cup} \mathcal{USAT}.$$

Resolution and DP-reduction

Clauses C, D are **resolvable** if $C \cap \overline{D} = \{x\}$:

$$\mathbf{C \diamond D} := (C \setminus \{x\}) \cup (D \setminus \{\overline{x}\}).$$

DP-reduction (or "variable elimination"):

$$\mathbf{DP_v(F)} := \{C \in F : v \notin \operatorname{var}(C)\} \cup \\ \{\, C \diamond D : C, D \in F \wedge C \cap \overline{D} = \{v\} \,\}.$$

$\mathrm{DP}_v(F)$ is semantically the existential quantification of F by v:

$$\mathrm{DP}_v(F) \longleftrightarrow \exists\, v : F.$$

Singular variables

A variable v is **singular** for F if

- in one sign it occurs only once,
- while in the other sign it occurs at least once.

Singular DP-reduction ("sDP-reduction") is $F \rightsquigarrow \mathrm{DP}_v(F)$ for a singular variable.

sDP-reduction decreases the number of clauses at least by one.

TOC: MU and its structures

- MU, SMU
- deficiency
- MU(1)
- non-singularity
- MU(2)

See Kleine Büning and Kullmann [5].

MU and SMU

Minimal unsatisfiability: no clause can be removed.

$$\boldsymbol{\mathcal{MU}} := \{F \in \mathcal{USAT} \mid \forall\, C \in F : F \setminus \{C\} \in \mathcal{SAT}\}.$$

Saturated minimal unsatisfiability: no literal can be added.

$$\boldsymbol{\mathcal{SMU}} := \{F \in \mathcal{MU} \mid \forall\, C \in F \;\forall\, C' \supset C : (F \setminus \{C\}) \cup \{C'\} \in \mathcal{SAT}\}.$$

Lemma

$F \in \mathcal{SMU}$ iff for all $v \in \operatorname{var}(F)$ and $\varepsilon \in \{0,1\}$ holds $\langle v \to \varepsilon \rangle * F \in \mathcal{MU}$.

Deficiency

The basic **complexity parameter** of $F \in \mathcal{MU}$ is the **deficiency**:

$$\begin{aligned} \boldsymbol{\delta}(\mathbf{F}) &:= c(F) - n(F) \\ c(F) &:= |F| \\ n(F) &:= |\mathrm{var}(F)|. \end{aligned}$$

The basic fact:

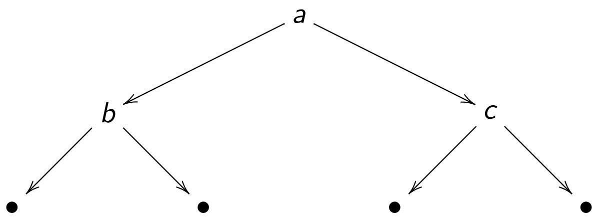
$$F \in \mathcal{MU} \Longrightarrow \delta(F) \geq 1.$$

We use here $\boldsymbol{\mathcal{MU}}(\mathbf{k}) := \{F \in \mathcal{MU} : \delta(F) = k\}$.

MU(1)

Timeline:

- Aharoni and Linial [1] (1986) $\mathcal{SMU}(1)$
- Davydov, Davydova, and Kleine Büning [3] (1998) $\mathcal{MU}(1)$
- Kullmann [6] (2000) tree model



1. $\{\{a,b\},\{a,\overline{b}\},\{\overline{a},c\},\{\overline{a},\overline{c}\}\} \in \mathcal{SMU}(1)$. Singular: b, c.
2. $\{\{a,b\},\{\overline{b}\},\{\overline{a},c\},\{\overline{a},\overline{c}\}\} \in \mathcal{MU}(1)$. Singular: all.
3. $\{\{a,b\},\{\overline{b}\},\{\overline{a},c\},\{\overline{c}\}\} \in \mathcal{MMU}(1)$ ("marginal").

Eliminating trivialities

We consider sDP-reduction as eliminating "trivialities".

- So all of $\mathcal{MU}(1)$ boils down to $\{\bot\}$.
- Intuitively one can understand applying sDP as removing some "MU(1)-hunch".

A clause-set F is called **non-singular** if it does not contain singular variables.

- The set of singular variables of F is $\mathbf{var_s(F)} \subseteq \mathrm{var}(F)$.
- So F is nonsingular iff $\mathrm{var_s}(F) = \emptyset$.

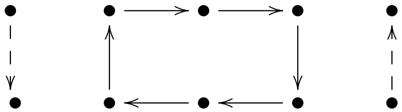
$$\boldsymbol{\mathcal{MU}'} := \{F \in \mathcal{MU} : \mathrm{var_s}(F) = \emptyset\}.$$

MU(2)

A breakthrough was achieved by Kleine Büning [4] (2000).
The elements of $\mathcal{MU}'(2) = \mathcal{SMU}(2)$ are **precisely** the following clause-sets for $n \geq 2$:

$$x_1 \to x_2,\ x_2 \to x_3, \dots, x_{n-1} \to x_n,\ x_n \to x_1$$
$$\{x_1, \dots, x_n\},$$
$$\{\overline{x_1}, \dots, \overline{x_n}\}.$$

That is, one cycle, with opposed forced "directions" ($n = 6$):



Understanding UNSAT — understanding MU

For unsatisfiable $F \in \mathcal{USAT}$

we want to **"understand"** its unsatisfiability.

We want to SEE it.

Each $F' \subseteq F$ with $F' \in \mathcal{MU}$ contains one explanation:

- The additional clauses in $F \setminus F'$ can make the contradiction of F' more easily accessible, but do not contribute "another reason".
- Different F' provide different explanations.

So we have to "explain" unsatisfiability for $F \in \mathcal{MU}$.

Explain $\mathcal{MU}'(k)$ for $k = 1, 2, 3, \ldots$.

The Classification Conjecture

Conjecture *For every $k \in \mathbb{N}$ there are finitely many "patterns", which explain $\mathcal{MU}'(k)$ completely.*

1. $\mathcal{MU}'(1) = \{\bot\}$
2. $\mathcal{MU}'(2) =$ cycles of length $n \geq 2$ plus "at least one variable is false" plus "at least one variable is true".
3. $\mathcal{MU}'(3)$: in preparation.

Two problems

For $F \in \mathcal{MU}$ let

$$\mathbf{sDP(F)} := \{F' \in \mathcal{MU}' : F \xrightarrow{\text{sDP}}_* F'\}.$$

First problem: We want to understand $F \in \mathcal{MU}(k)$. For that we consider $\text{sDP}(F)$. Note $\text{sDP}(F) \subset \mathcal{MU}(k)$.

Easiest is $|\text{sDP}(F)| = 1$ — when does this hold?

Second problem: A transition $F \xrightarrow{\text{sDP}}_* F' \in \mathcal{MU}'$ removes "trivialities" — what if we have to understand these trivialities?

The "singularity index"

Theorem

For $F \in \mathcal{MU}$ and $F', F'' \in \mathrm{sDP}(F)$ we have $n(F') = n(F'')$.

This allows us to define the **singularity index si(F)** $\in \mathbb{N}_0$ as $\mathrm{si}(F) := n(F) - n(F')$ for some $F' \in \mathrm{sDP}(F)$.

Corollary

If $F \in \mathcal{MU}(2)$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Here $F' \cong F''$ denotes isomorphism.

Confluence

Theorem

If $F \in \mathcal{SMU}$, *then* $|\mathrm{sDP}(F)| = 1$.

Confluence modulo isomorphism

Theorem

If for $F \in \mathcal{MU}$ we have $\mathrm{sDP}(F) \subseteq \mathcal{SMU}$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Corollary

If $F \in \mathcal{MU}(2)$, then for $F', F'' \in \mathrm{sDP}(F)$ we have $F' \cong F''$.

Commutativity

In Kullmann and Luckhardt [7, 8] it is shown:

Lemma

For $F \in \mathcal{CLS}$, variables $v_1, \dots, v_k$ and a permutation $\pi \in S_k$ we have that

$$\mathrm{DP}_{v_1,\dots,v_k}(F), \quad \mathrm{DP}_{v_{\pi(1)},\dots,v_{\pi(k)}}(F)$$

are equal modulo subsumption.

Controlled permutations

- The key is to exchange neighbouring sDP-reductions such that we again obtain an sDP-reduction.
- Distinguish between 1-singular variables (occurring in both signs only once) and non-1-singular variables.

Conclusions

Contributions:

- For minimally unsatisfiable clause-sets the number of sDP-reductions until reaching nonsingularity does not depend on the choice of reductions.
- Confluence of sDP-reduction for saturated minimally unsatisfiable clause-sets.
- Confluence modulo isomorphism of sDP-reduction in case all maximal sDP-reductions yield saturated minimally unsatisfiable clause-sets.
- Obtaining unique normalforms (up to isomorphism) for $\mathcal{MU}(2)$ via sDP-reduction.

Next steps:

- Fuller understanding of sDP-reduction.
- Generalisation to all of $\mathcal{CLS}$.

End

(references on the remaining slides).

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