Topoi: Theory and Applications

Oliver Kullmann

\(^1\)Computer Science, Swansea University, UK
http://cs.swan.ac.uk/~csoliver

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I treat “topos theory” as a theory, whose place is similar to, say, group-theory in relation to semigroup-theory:

1. A “topos” is a special category.
2. Namely a category with “good” “algebraic” structures.
3. Similar to the operations that can be done with finite sets (or arbitrary sets).
4. A “Grothendieck topos” is a special topos, having more “infinitary structure”.
5. It is closer to topology.
“Topos” is Greek, and means “place”.

Its use in mathematics likely is close to “space”, either “topological space” or “set-theoretical space”.

Singular “topos”, plural “topoi” (“toposes” seems to be motivated by a dislike for Greek words).

Concept invented by Alexander Grothendieck.

What was original a “topos”, became later a “Grothendieck topos”.

Grothendieck topoi came from algebraic geometry: “a “topos” as a “topological structure”.

Giraud (student of Grothendieck) characterised categories equivalent to Grothendieck topoi.
Then came “elementary topoi”, perhaps more motivated from set theory.

William Lawvere (later together with Myles Tierney) developed “elementary” (first-order) axioms for Grothendieck topoi.

The “subobject classifier” plays a crucial role here.

Elementary topoi generalise Grothendieck topoi.

Nowadays it seems “topos” replaces “elementary topos”.

The subobject classifier plays a crucial role here.
Outline

Introduction
Grothendieck topoi
Topoi
  Exponentiation
  Characteristic maps
The category of sets
The topos of presheaves
  Monoid operations
  Directed graphs
  Generalisations
Comma categories
Properties
Generalised topology

From [Borceux, 1994]:

- A **Grothendieck topos** is a category equivalent to a category of sheaves on a site.
- A Grothendieck topos is complete and cocomplete.
- Every Grothendieck topos is a topos.
- A topos in general is only finitely complete and finitely cocomplete.
Three operations with sets

Three related properties of the category $\mathbf{SET}$:

**Function sets** For sets $A, B$ we have the set $B^A$ of all maps $f : A \to B$.

**Characteristic maps** For set $A$ the subsets are in 1-1 correspondence to maps from $A$ to $\{0, 1\}$.

**Powersets** For a set $A$ we have the set $\mathbb{P}(A)$ of all subsets of $A$.

Via characteristic maps we get powersets from function spaces:

$$\mathbb{P}(A) \cong \{0, 1\}^A.$$

- Perhaps in categories which have “map objects” and “characteristic maps”, we have also “power objects”?
- Conversely, perhaps from power objects we get, as in set theory, map objects and characteristic maps?
Exponentiation: The idea

Fix a category \( \mathcal{C} \). We consider “exponentiation” with an object \( E \) — easiest to fix \( E \):

\[
pow_E : B \in \text{Obj}(\mathcal{C}) \mapsto B^E \in \text{Obj}(\mathcal{C}).
\]

What could be the universal property? “Currying”?!:

\[
\text{Mor}(A \times E, B) \cong \text{Mor}(A, B^E).
\]

So we need products in \( \mathcal{C} \). Looks like adjoints?! Recall \( F : \mathcal{C} \to \mathcal{D}, G : \mathcal{D} \to \mathcal{C} \) yields an adjoint \( (F, G) \) iff there is a natural isomorphism

\[
\text{Mor}(F(A), B) \cong \text{Mor}(A, G(B)).
\]

So \( F(A) := A \times E \) and \( G(B) := pow_E(B) \).
Definition

The category \( \mathcal{C} \) has exponentiation with power \( E \in \text{Obj}(\mathcal{C}) \) if the functor \( A \in \text{Obj}(\mathcal{C}) \mapsto A \times E \in \text{Obj}(\mathcal{C}) \) has a right adjoint \( B \in \text{Obj}(\mathcal{C}) \mapsto B^E \in \text{Obj}(\mathcal{C}) \).

According to the general theory of adjoints, this is equivalent to the property that for all \( B \in \text{Obj}(\mathcal{C}) \) the functor \( P := (A \times E)_{A \in \text{Obj}(\mathcal{C})} \) has a universal arrow ("cofree object", "coreflection") from \( P \) to \( B \), that is, a morphism

\[
e : B^E \times E \to B
\]

such that for all \( e' : A \times E \to B \) there exists a unique \( f : A \to B^E \) with \( e' = e \circ (f \times \text{id}_E) \).
Exponentiation: The formulation II

Definition
A category $\mathcal{C}$ with binary products has exponentiation, if all powers admit exponentiation.

- If $\mathcal{C}$ has exponentiation, then so does its skeleton, and thus having exponentiation is an invariant under equivalence of categories.
- For exponentiation we need the existence of binary products, however, as usual, a different choice of binary products leads to (correspondingly) isomorphic exponentiations.
- So the above “with” can be interpreted in the weak sense, just sheer existence is enough (no specific product needs to be provided).
Definition
A category is **cartesian-closed**, if it has finite limits and exponentiation.

- Being cartesian-closed is just a property of categories, no additional structure is required (“has” means “exists”).
- If a category is cartesian-closed, so is its skeleton, and thus being cartesian-closed is an invariant under equivalence of categories.

The category $\mathbf{Cat}$ of all (small) categories is cartesian-closed, with $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$ as the exponential object of $\mathcal{C}, \mathcal{D}$, and so we can write $\mathcal{D}^\mathcal{C} := \mathbf{Fun}(\mathcal{C}, \mathcal{D})$. 
Subobject classifier: The idea

Now let’s turn to “characteristic functions”. Consider a category $\mathcal{C}$ with finite products and an object $X$.

A subobject $A$ of $X$ shall correspond to that morphism $\chi_A : X \to \Omega$

for some fixed “subobject classifier” $\Omega \in \text{Obj}(\mathcal{C})$,

such that the image of $A$ under $\chi_A$

is the same as $t : 1 \to \Omega$

for some fixed $t$.

- If such a pair $(\Omega, t)$ exists, it is called a “subobject classifier” of $\mathcal{C}$.
- So consider a subobject(-representation) $i : A \hookrightarrow X$. 
Subobject classifier: The conditions

We get the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{1_A} & 1 \\
\downarrow i & & \downarrow t \\
X & \xrightarrow{\chi_A} & \Omega
\end{array}
\]

What are the conditions?

1. For every mono \( i : A \hookrightarrow X \) there shall be exactly one \( \chi_A : X \to \Omega \) making the diagram commute, and fulfilling the further conditions.

2. A pushout?

3. No, a pullback!
Definition

For a category \( \mathcal{C} \) with a terminal object \( 1_{\mathcal{C}} \), a subobject classifier is a pair \((\Omega, t)\) with \( \Omega \in \text{Obj}(\mathcal{C}) \) and \( t : 1_{\mathcal{C}} \to \Omega \), such that for all monos \( i : A \hookrightarrow X \) there is exactly one \( \chi_A : X \to \Omega \) such that \((i, 1_A)\) is a pullback of \((t, \chi_A)\).

- All subobject classifiers for \( \mathcal{C} \) are pairwise isomorphic.
- If \( \mathcal{C} \) has a subobject classifier, then so does its skeleton, and thus having a subobject classifier is an invariant under equivalence of categories.
Definition of ("elementary") topoi

Definition

A **topos** is a cartesian-closed category which has a subobject classifier.

- Being a topos is just a property of categories, no additional structure is required ("has" means "exists").
- If a category is a topos, so is its skeleton, and thus being a topos is an invariant under equivalence of categories.
Equivalent characterisation of topoi

With [Lane and Moerdijk, 1992], Section IV.1:

- The “global” point of view is used, similar to the use of adjoints in characterising exponential objects.
- The “power object” operation $\mathbb{P}$ is assumed.
- This is a map $\mathbb{P} : \text{Obj}(\mathcal{C}) \to \text{Obj}(\mathcal{C})$ such that for all objects $A, B \in \text{Obj}(\mathcal{C})$ there are natural isomorphisms

$\text{Sub}(A \times B) \cong \text{Mor}(A, \mathbb{P}(B))$

(between sets).

Using $\Omega := \mathbb{P}(1)$ we get the subobject classifier.
Reminder: limits

- The category of sets is complete, that is, has all (small) limits.
- It is also cocomplete (has all (small) colimits), but we do not need this here (finite cocompleteness follows from being a topos).
- Completeness is equivalent to having all (small) products and all (binary) equalisers.
- The canonical terminal object is the empty product, i.e., $\prod \emptyset = \emptyset^\emptyset = \mathcal{P}(\emptyset) = \{\emptyset\} = \{0\} = 1$.

$$\left( X_i \right)_{i \in I} \xrightarrow{\text{pr}_p} \prod_{i \in I} X_i \xrightarrow{\text{pr}_q} \cdots X_p \xleftarrow{\cdots} \cdots X_q \cdots$$

$$X \xrightarrow{f} Y \xleftarrow{\text{in}} \{ x \in X : f(x) = g(x) \} \xrightarrow{\text{in}} X \xrightarrow{f} Y$$
Reminder: pullbacks

\[(a, b) \in A \times B : f(a) = g(b)\]
Exponentiation and subobject classifier

Exponentiation:

\[(X, Y) \mapsto Y^X := \{f : X \to Y\}\]

\[e : Y^X \times X \to Y, \quad e(f, x) := f(x).\]

Subobject classifier:

\[\Omega := \{0, 1\} = \mathbb{P}(1)\]

\[t : 1 = \{0\} \hookrightarrow \Omega, \quad 0 \mapsto 1.\]

\[\mathbb{P}(X) \hookrightarrow \Omega^X, \quad A \mapsto \chi_X(A) := A \times \{1\} \cup (X \setminus A) \times \{0\}\]
The category of finite sets

- The full subcategory of $\mathcal{C} \subseteq \mathcal{C}$ given by all finite sets (in the current universe, of course) is a topos, with the same operations.
- In general, if for a topos $\mathcal{C}$ and a full subcategory $\mathcal{C}$
  - $\mathcal{C}$ is closed under the the topos-operations (finite product, exponentiation, subobject classifier),
  - $\mathcal{C}$ is closed under subobject-formation,
then also $\mathcal{C}$ is a topos.
Monoid operations

Consider a fixed monoid $M = (M, \cdot, 1)$. An operation of $M$ on $X$ is given by a map

$$\ast : M \times X \to X$$

such that for all $a, b \in M$ and $x \in X$ we have

$$1 \ast x = x$$

$$a \ast (b \ast x) = (a \cdot b) \ast x.$$

- $(M, \ast)$ is also called an $M$-set.
- Isomorphically, we have the point of view of a “representation via transformations”: a morphism from $M$ into the transformation monoid

$$\mathfrak{S}(X) = (X^X, \circ, \text{id}_X).$$

See Section 4.6 in [Goldblatt, 2006] for basic information on the topos of $M$-sets.
The category of $M$-sets

For $M$-sets $X, Y$, a morphism $f : X \rightarrow Y$ is a map fulfilling

$$\forall a \in M \forall x \in X : f(a \ast x) = a \ast f(x).$$

The category of $M$-sets is denoted by $\text{OPR}_M(\text{SET})$.

- I use the terminological distinction between “action” and “operation”, where for the former structure on the object acted upon is involved (e.g., the action of a set on a group via automorphisms), and for the latter structure on the side of acting object (the operation of a group on a set).

- $\text{OPR}_M(\text{SET})$ is a concrete category.

$\text{OPR}_M(\text{SET})$ is canonically isomorphic to the functor category $\text{SET}^M$, considering $M$ as a one-object category.
Some remarks

1. The functor $O : \text{MON} \to \text{CAT}'$, mapping monoid $M$ to category $\text{OPR}_M(\text{SET})$, is a contravariant functor.

2. Here $\text{CAT}'$ is the category of “large” categories (in the parameter-universe).

3. More generally, the functor $O : \text{MON} \times \text{CAT}' \to \text{CAT}'$, given by $(M, \mathcal{C}) \mapsto \mathcal{C}^M$, mapping a monoid $M$ and a category $\mathcal{C}$ to the category of operations of $M$ on $\mathcal{C}$, is a bifunctor, contravariant in the first argument.

4. More generally, the mapping $\text{FUN} : \text{CAT} \times \text{CAT}' \to \text{CAT}'$, $(\mathcal{C}, \mathcal{D}) \mapsto \mathcal{D}^\mathcal{C}$, is a bifunctor, contravariant in the first argument.

5. More generally, for a cartesian-closed category $\mathcal{C}$, the mapping $\mathcal{C}^2 \to \mathcal{C}, (X, Y) \mapsto Y^X$, is a bifunctor, contravariant in the first argument.
The forgetful functor \( V : \text{OPR}_M(\text{SET}) \) has a left-adjoint, the formation of free operations. So \( V \) preserves limits.  

- I.e., if limits exist, they must have the underlying sets as given by the limits in \( \text{SET} \).

- It is easy to see, as in all algebraic categories, that the operations of \( M \) defined in the obvious ways for the \( \text{SET} \)-limits, yield limits in \( \text{OPR}_M(\text{SET}) \).

It also follows that the monomorphisms of \( \text{OPR}_M(\text{SET}) \) are precisely the injective morphisms.
Representable functors

The representable functors of a category $\mathcal{C}$ are those functors $F : \mathcal{C} \to \mathcal{S}ET$ which are isomorphic to a Hom-functor $X \in \text{Obj } \mathcal{C} \mapsto \text{Mor}(A, X) \in \text{Obj}(\mathcal{S}ET)$ for some $A \in \text{Obj}(\mathcal{C})$ (the representing object).

- We consider the objects of $\mathcal{OPR}_M(\mathcal{S}ET)$ as functors (“covariant presheaves”).
- There is then only one object, thus only one Hom-functor.
- This is the canonical operation of $M$ on itself, via multiplication.
For \( M \)-sets \( B, E \) the exponential \( B^E \) is defined as having

- base set \( \text{Mor}(M \times E, B) \)
- operation (for \( m \in M \) and a morphism \( f : M \times E \to B \))
  \[
  (m \ast f)(a, e) := f(m \cdot a, e)
  \]
- evaluation \( e : B^E \times E \to B \) given by
  \[
  e(f, e) := f(1, e).
  \]
Alternative for groups

If $M$ is a group, then we have a simple (of course, isomorphic) possibility to define the exponential $B^E$:

- base set $\text{Mor}_{\text{SET}}(E, B)$
- operation (for $g \in M$ and a map $f : E \to B$)
  \[(g \star f)(e) := g \star f(g^{-1} \star e)\]
- evaluation $e : B^E \times E \to B$ given by
  \[e(f, e) := f(e).\]
Lemma

Consider a category $\mathcal{C}$, an object $A \in \text{Obj}(\mathcal{C})$ and a functor $T : \mathcal{C} \to \mathbf{SET}$. The Yoneda map

$$Y_{A,T} : \text{NAT}(\text{Mor}_\mathcal{C}(A, -), T) \to T(A)$$

is a bijection.

So, for $\mathcal{C} = \text{OPR}_M(\mathbf{SET})$, for every $M$-set $X$ we have a natural bijection

$$\text{Mor}(M, X) \cong X.$$

This is also easy to see directly, since a morphism from $M$ to $X$ is uniquely determined by the image of 1 ($M$ is the free operation generated by one element).
Subobject classifier

What shall be $\Omega$ ?!
Let’s consider the $M$-set $M$ and its subobjects:

1. Subobjects are the left ideals of the semigroup $M$ (subsets stable under left multiplication).
2. As we have seen, $\text{Mor}(M, \Omega) \cong \Omega$ holds.

So we should take as the base set of $\Omega$ the set of left ideals of $M$:

$$\Omega := \{ I \subseteq M \mid \forall a \in M \forall x \in I : a \cdot x \in I \}.$$  
(Thus $|\Omega| \geq 2$.) It is natural to choose $t := M \in \Omega$.

What is now the operation of $M$ on $\Omega$ ?

$$a \ast \omega := \{ b \in M : b \cdot a \in \omega \}$$

for $\omega \in \Omega$ and $a \in M$. 

Lemma

For an $M$-set $X$ and a morphism $f : X \to \Omega$ we have

$$\forall x \in X : f(x) = \{ a \in M : f(a \ast x) = M \}.$$  

For a subset $A \subseteq X$ there exists a morphism $f : X \to \Omega$ with $f^{-1}(\{M\}) = A$ iff $A$ is closed (i.e., is a subspace), in which case $f$ is unique, namely $f = \chi_A$ with

$$\chi_A(x) := \{ a \in M : a \ast x \in A \}.$$

Proof: For a morphism $f : X \to \Omega$ we have:

$$f(a \ast x) = M \iff a \ast f(x) = M \iff$$

$$\{ b \in M : b \cdot a \in f(x) \} = M \iff a \in f(x).$$

$M$ operates trivially on $M \in \Omega$, so $f^{-1}(\{M\})$ is closed. Finally $a \ast \chi_A(x) = \{ b \in M : b \cdot a \in \chi_A(x) \} = \{ b \in M : (b \cdot a) \ast x \in A \} = \{ b \in M : b \ast (a \ast x) \in A \} = \chi_A(a \ast x)$. □
General directed graphs
With Remark B:2.3.19 in [Johnstone, 2002]:

- If $\mathcal{C}$ is a finite category and $\mathcal{D}$ is a topos, then $\mathcal{D}^\mathcal{C}$ is a topos.
- If $\mathcal{C}$ is a small category and $\mathcal{D}$ is a cocomplete topos, then $\mathcal{D}^\mathcal{C}$ is a topos.

For a small category $\mathcal{C}$
the category $\mathcal{SET}^\mathcal{C}$ of **presheaves**
is a topos.
Reminder: Comma categories

See

http://en.wikipedia.org/wiki/Comma_category

for more information.

Consider functors $F : \mathcal{A} \to \mathcal{C}$, $G : \mathcal{B} \to \mathcal{C}$. The **comma category** $(F \downarrow G)$ is defined as follows:

1. objects are triples $(a, b, \varphi)$, where $a \in \mathcal{A}$, $b \in \mathcal{B}$, and $\varphi : F(a) \to G(b)$
2. morphisms $f : (a, b, \varphi) \to (a', b', \varphi')$ are pairs $f = (\alpha, \beta)$, where $\alpha : a \to a'$, $\beta : b \to b'$, and $\varphi' \circ F(\alpha) = G(\beta) \circ \varphi$.

Special cases:

- An object $X$ of a category $\mathcal{C}$ stands for $1 \mapsto X$.
- A category $\mathcal{C}$ stands for $\text{id}_\mathcal{C}$.
- $(\mathcal{C} \downarrow X)$ also written as “$\mathcal{C}/X$” (“slice category”).
- $(\mathcal{C} \downarrow G)$ also written as “$\mathcal{C}/G$” (“Artin glueing”).
Comma categories yielding topoi

Consider topoi $\mathcal{B}, \mathcal{C}$ and a functor $G : \mathcal{B} \to \mathcal{C}$.

**Theorem [Wraith 1974, [Carboni and Johnstone, 1995]]:**

*If $G$ preserves pullbacks, then $(\mathcal{C} \downarrow G)$ is also a topos.*

Special cases:

1. The product of two topoi is a topos.
2. Slices of a topos are topoi.
The topos of (labelled, generalised) clause-sets

Consider a fixed monoid $M$. Consider the (forward) powerset functor

$$P_f : \text{OPRM}(\mathsf{SET}) \to \mathsf{SET}$$

which

- maps an $M$-set $X$ to $P_f(X)$,
- maps $f : X \to Y$ to $P_f(f) : P_f(X) \to P_f(Y)$, where
  
  $$P_f(f)(S) := f(S).$$

Now let

$$\mathsf{LCLS}_M := (\mathsf{SET} \downarrow P_f)$$

$P_f$ does not preserve pullbacks, nevertheless these categories are topoi.

- $\mathsf{LCLS}_{\{1\}}$ is the category of labelled hypergraphs
- $\mathsf{LCLS}_{\mathbb{Z}_2}$ is the category of labelled clause-sets (allowing degenerated and non-polarised literals!).

Without the labelling, we obtain quasi-topoi.
Actually, [Carboni and Johnstone, 1995] show for topoi $\mathcal{B}$, $\mathcal{C}$ and a functor $G : \mathcal{B} \to \mathcal{C}$:

$G$ preserves pullbacks if and only if $(\mathcal{C} \downarrow G)$ is a topos.

Now $G$ clearly does not preserve pullbacks, however we have a topos ...
Factorisation

Consider a category $\mathcal{C}$ and a morphism $f : A \to B$.

- $f$ has an **epi-mono factorisation** if
  \[
  f = i \circ \pi
  \]
  for some epimorphism $\pi : A \to C$ and some monomorphism $i : C \to B$.

- Such a factorisation is **unique** if for every other epi-mono factorisation $f = i' \circ \pi'$, $i' : A \to C'$, $\pi' : C' \to B$, there is an isomorphism $\varphi : C \to C'$ with commutative diagram:

```
    C
   / \  \\
  i   \   \n  / \  |   |  \n A  \  |   |   \n  \  |   |   v
   \ |   |   B
      \|   |   \n      π |   \   |
      /   v   v
     \   C'   C'
```

- Consider a category $\mathcal{C}$ and a morphism $f : A \to B$.
  - $f$ has an **epi-mono factorisation** if
    \[
    f = i \circ \pi
    \]
    for some epimorphism $\pi : A \to C$ and some monomorphism $i : C \to B$.
  - Such a factorisation is **unique** if for every other epi-mono factorisation $f = i' \circ \pi'$, $i' : A \to C'$, $\pi' : C' \to B$, there is an isomorphism $\varphi : C \to C'$ with commutative diagram:

```
    C
   / \  \\
  i   \   \n  / \  |   |  \n A  \  |   |   \n  \  |   |   v
   \ |   |   B
      \|   |   \n      π |   \   |
      /   v   v
     \   C'   C'
```
Topoi have unique factorisations

- A category **has epi-mono factorisation** (also “epi-mono decomposition”, or just “factorisation” or “decomposition”) if every morphism has an epi-mono factorisation.
- And similarly one says a category **has unique epi-mono factorisation**.

**Lemma**

*A topos has unique epi-mono factorisation.*
Topoi are balanced

Recall:
- A \textbf{bimorphism} is a morphism which is epi and mono.
- A category is \textbf{balanced} if every bimorphism is iso.

\textbf{Lemma}

\textit{Every category with unique factorisation is balanced.}

\textbf{Proof:} Consider a bimorphism $f : A \to B$.

\begin{center}
\begin{tikzpicture}
  \node (A) at (0,0) {$A$};
  \node (B) at (0,-2) {$B$};
  \node (idA) at (-1,-1) {$A$};
  \node (idB) at (1,-1) {$B$};
  \node (phi) at (0,-1) {$\varphi$};

  \draw[->] (A) to node [left] {$\text{id}_A$} (idA);
  \draw[->] (B) to node [right] {$\text{id}_B$} (idB);
  \draw[->] (A) to node [above] {$f$} (B);
  \draw[->] (B) to node [below] {$f$} (idB);
  \draw[->] (idA) to node [below] {$\varphi$} (phi);
  \draw[->] (phi) to node [above] {$\varphi$} (idB);
\end{tikzpicture}
\end{center}
Topoi have power objects

The (global) “power object map” can also be localised.
Lemma

Every topos is finitely cocomplete.

Proof: (not completely trivial)
With [Lane and Moerdijk, 1992], Section IV.8:

**Lemma**

*For every object $A$ in a topos, the partial order $\text{Sub}(A)$ of subobjects is a Heyting lattice.*
Internal Heyting algebras

With [Lane and Moerdijk, 1992], Section IV.8:

**Lemma**

For every object $A$ in a topos, the power object $P(A)$ can be given the structure of an Heyting algebra object (an “internal Heyting algebra”). In particular, this applies for the subobject classifier $\Omega = P(1)$.

For each object $X$ the internal structure of $P(A)$ makes $\text{Mor}(X, P(A))$ a Heyting algebra. Now the canonical bijection between $\text{Sub}(X \times A)$ and $\text{Mor}(X, P(A))$ becomes an isomorphism of Heyting algebras.

**Proof:** Conjunction $\wedge : \Omega \times \Omega \to \Omega$ is the characteristic morphism of $1 \to \Omega \times \Omega$. ...
Summary

I  The notion of a “topos” has been defined.

II Examples via categories of presheaves and comma categories have been discussed.

III Basic elementary properties of topoi have been presented.
Bibliography I


End