The adjoint functor theorem

Oliver Kullmann[1]

[1]Computer Science, University of Wales Swansea, UK
`http://cs.swan.ac.uk/~csoliver`

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Introduction

The main topic is:

- after a review of notions and theorems relevant to adjunctions
- we prove the "adjoint functor theorem", a necessary and sufficient criterion for the existence of (left- or right-) adjoints of a given functor.

With this criterion the construction of adjoint functors is often considerably facilitated.

We will also outline main corollaries and main applications / examples.

The adjoint functor theorem

Theorem *Consider a complete category $\mathfrak{C}$ with small morphism-sets, and a functor $V : \mathfrak{C} \to \mathfrak{S}$.*

Then V has a left adjoint if and only if V respects all limits (is "continuous"), and for every every $X \in \mathrm{Obj}(\mathfrak{S})$ we have a "solution set", that is, a family $(A_i)_{i \in I}$ of objects in $\mathfrak{C}$ together with a family $(f_i)_{i \in I}$ of morphisms $f_i : X \to V(A_i)$ in $\mathfrak{S}$ such that

for all $A \in \mathrm{Obj}(\mathfrak{C})$ and all $f : X \to V(A)$ there is some $i \in I$ together with some morphism $g : A_i \to A$ in $\mathfrak{C}$ with

$$f = V(g) \circ f_i.$$

(Remark: For $|I| = 1$ we nearly get the condition, that A is universal for X to V (uniqueness is missing).)

Overview

1. Review
2. Finding initial and terminal objects
3. The adjoint functor theorem
4. Reflections and coreflections
5. The fundamental examples
6. Topological examples

Cancellation and inversion

Consider a category $\mathfrak{C}$ and a morphism f in $\mathfrak{C}$:

- Regarding cancellation:
  - f is a **monomorphism** if f is left-cancellable.
  - f is an **epimorphism** if f is right-cancellable.
- Regarding invertibility:
  - f is a **coretraction** if f is left-invertible.
  - f is a **retraction** if f is right-invertible.
- It follows, that f is an **isomorphism** iff f is a retraction and a coretraction.
- Necessary conditions for cancellability:
  - Every coretraction is a monomorphism
  - Every retraction is an epimorphism.
- The following conditions are equivalent:
  - f is an isomorphism.
  - f is a monomorphism and a retraction.
  - f is an epimorphism and a coretraction.

Bimorphisms

- A morphism is called a **bimorphism** if it is a monomorphism and an epimorphism.
- Every isomorphism is a bimorphism.

A category is called **balanced** if every bimorphism is an isomorphism.

- In a balanced concrete category, the isomorphisms are exactly the bijections.
- A concrete category, where every bijection is an isomorphism, does not need to be balanced, as $\mathfrak{MON}$ shows (the canonical injection $j : (\mathbb{Z}, \cdot, 1) \to (\mathbb{Q}, \cdot, 1)$ is a bimorphism).
  (The category of groups, on the other hand, is balanced.)

If $V : \mathfrak{C} \to \mathfrak{S}$ is faithful, and $\mathfrak{C}$ is balanced, then V is discrete (i.e., the fibres of V are discrete orders).

Equalisers and coequalisers

Consider two morphisms $f, g : X \to Y$.

- $\alpha : E \to X$ is an **equaliser** for f, g if $f \circ \alpha = g \circ \alpha$, and for every $\alpha' : E' \to X$ with $f \circ \alpha' = g \circ \beta'$ there is a unique $\beta : E' \to E$ with $\alpha' = \alpha \circ \beta$.
- $\alpha : Y \to C$ is an **coequaliser** for f, g if $\alpha \circ f = \alpha \circ g$, and for every $\alpha' : Y \to C$ with $\alpha' \circ f = \alpha' \circ g$ there is a unique $\beta : E \to E'$ with $\alpha' = \beta \circ \alpha$.

We have

1. Every equaliser is a monomorphism.
2. Every coequaliser is an epimorphism.

To see this, let α be an equaliser of f, g, and consider φ, ψ with $\alpha \circ \varphi = \alpha \circ \psi$. Trivially $f(\alpha\varphi) = g(\alpha\varphi)$, and the (unique(!)) β with $\alpha\beta = \alpha\varphi$ is $\beta = \varphi$. Now $\alpha\psi = \alpha\varphi$, and thus $\psi = \beta = \varphi$. $\surd$

Upon an object

Let $V : \mathfrak{C} \to \mathfrak{S}$ be a functor, and $X \in \mathrm{Obj}(\mathfrak{S})$. The **comma-category $(X \downarrow V)$** has

- objects (C, α) where $C \in \mathrm{Obj}(\mathfrak{C})$ and $\alpha : X \to V(C)$;
- morphisms from (C, α) to (D, β) are $f : C \to D$ with commuting

$$\begin{tikzcd} V(C) \arrow[rr, "V(f)"] & & V(D) \\ & X \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \end{tikzcd}$$

- identities and composition as in $\mathfrak{C}$; check the commutativity of

$$\begin{tikzcd} V(C) \arrow[rr, "V(\mathrm{id}_C)"] & & V(C) \\ & X \arrow[ul, "\alpha"] \arrow[ur, "\alpha"'] & \end{tikzcd}, \quad \begin{tikzcd} V(C) \arrow[r, "V(f)"] & V(D) \arrow[r, "V(g)"] & V(E) \\ & X \arrow[ul, "\alpha"] \arrow[u, "\beta"] \arrow[ur, "\gamma"'] & \end{tikzcd}$$

So $(X \downarrow V)$ is a concrete category over $\mathfrak{C}$ via the projection $P : (X \downarrow V) \to \mathfrak{C}$, $P((C, \alpha)) := C$.

The other side (into an object)

The **comma-category $(V \downarrow X)$** has

- objects (C, α) where $C \in \mathrm{Obj}(\mathfrak{C})$ and $\alpha : V(C) \to X$;
- morphisms from (C, α) to (D, β) are $f : C \to D$ with commuting

$$\begin{tikzcd} V(C) \arrow[rr, "V(f)"] \arrow[dr, "\alpha"'] & & V(D) \arrow[dl, "\beta"] \\ & X & \end{tikzcd}$$

- identities and composition as in $\mathfrak{C}$; check the commutativity of

$$\begin{tikzcd} V(C) \arrow[rr, "V(\mathrm{id}_C)"] \arrow[dr, "\alpha"'] & & V(C) \arrow[dl, "\alpha"] \\ & X & \end{tikzcd} , \quad \begin{tikzcd} V(C) \arrow[r, "V(f)"] \arrow[dr, "\alpha"'] & V(D) \arrow[r, "V(g)"] \arrow[d, "\beta"] & V(E) \arrow[dl, "\gamma"] \\ & X & \end{tikzcd}$$

So $(V \downarrow X)$ is a concrete category over $\mathfrak{C}$ via the projection $P : (X \downarrow V) \to \mathfrak{C}$, $P((C, \alpha)) := C$.

Universality via commas

Let $V : \mathfrak{C} \to \mathfrak{S}$ be a functor, and $X \in \mathrm{Obj}(\mathfrak{S})$.

- A **universal arrow** from X to V is an initial object of $(X \downarrow V)$.
- A **universal arrow** from V to X is a terminal object of $(V \downarrow X)$.

In [Borceux] the terminology "(co)reflection of X along V" is used; universal constructions were (under **the above** name) made well-known in the 1950s, and since then they carried this name.

The definitions carried out:

- A universal arrow from X to V is a pair (C, α) with $C \in \mathrm{Obj}(\mathfrak{C})$ and $\alpha : X \to V(C)$ such that for every (D, β) there is a unique $f : C \to D$ with $\beta = V(f) \circ \alpha$.
- A universal arrow from V to X is a pair (C, β) with $C \in \mathrm{Obj}(\mathfrak{C})$ and $\beta : V(C) \to X$ such that for every (D, α) there is a unique $f : C \to D$ with $\alpha = \beta \circ V(f)$.

Functorial morphisms

Given functors $F, G : \mathfrak{C} \to \mathfrak{D}$, a **natural transformation** $\eta : F \to G$ is a family $\eta = (\eta_X)_{X \in \mathrm{Obj}(\mathfrak{C})}$ with $\eta_X : F(X) \to G(X)$ such that for all objects A, B in $\mathfrak{C}$ and morphisms $f : A \to B$ the following diagram commutes:

$$\begin{tikzcd}
G(A) \arrow[r, "G(f)"] & G(B) \\
F(A) \arrow[u, "\eta_A"] \arrow[r, "F(f)"] & F(B) \arrow[u, "\eta_B"']
\end{tikzcd}$$

- Natural transformations $\eta : F \to G$ are morphisms between functors; the "vertical composition" of natural transformations yields the category $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ of functors from $\mathfrak{C}$ to $\mathfrak{D}$.
- The isomorphisms $\eta : F \hookrightarrow\!\!\!\!\rightarrow G$ in $\mathfrak{FUN}(\mathfrak{C}, \mathfrak{D})$ are exactly the natural transformations η which are "pointwise isomorphisms" $\eta_X : F(X) \hookrightarrow\!\!\!\!\rightarrow G(X)$.

Adjunctions via natural isomorphisms

Consider functors $V : \mathfrak{C} \to \mathfrak{S}$ and $F : \mathfrak{S} \to \mathfrak{C}$ between categories with small morphism-sets. We get two functors

$$\mathrm{Mor}(F(-_1), -_2) : \mathfrak{S}^{\mathrm{t}} \times \mathfrak{C} \to \mathfrak{SET}$$
$$\mathrm{Mor}(-_1, V(-_2)) : \mathfrak{S}^{\mathrm{t}} \times \mathfrak{C} \to \mathfrak{SET}.$$

(F, V) is called an **adjoint functor pair** if there is a natural isomorphism

$$\eta : \mathrm{Mor}(F(-_1), -_2) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mor}(-_1, V(-_2)).$$

- The triple (F, V, η) is called an **adjunction**, and η is called the **adjunction-isomorphism**.
- F is called a **left adjoint** of V, and V is called a **right adjoint** of F.
- V is called **right adjoint** (at all) if there exists some left adjoint F of V, while F is called **left adjoint** (at all) if there exists some right adjoint V of F.

Units

Consider an adjunction (F, V, η) with $V : \mathfrak{C} \to \mathfrak{S}$. For $X \in \mathrm{Obj}(\mathfrak{S})$ and $Y \in \mathrm{Obj}(\mathfrak{C})$ we get the (natural) bijection

$$\eta_{(X,Y)} : \mathrm{Mor}(F(X), Y) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mor}(X, V(Y)).$$

So for $Y := F(X)$ we get

$$\eta_{(X,F(X))} : \mathrm{Mor}(F(X), F(X)) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mor}(X, V(F(X))).$$

Thus

$$\varphi_X := \eta_{(X,F(X))}(\mathrm{id}_{F(X)}) : X \to V(F(X)).$$

- It follows, that $\varphi : \mathrm{id}_{\mathfrak{S}} \to V \circ F$ is a natural transformation, called the **unit** of (F, V, η).
- For every $X \in \mathrm{Obj}(\mathfrak{S})$ the pair $(F(X), \varphi_X)$ is a *universal arrow* from X to V.

Counits

For $X \in \mathrm{Obj}(\mathfrak{S})$ and $Y \in \mathrm{Obj}(\mathfrak{C})$ from the bijection

$$\eta_{(X,Y)} : \mathrm{Mor}(F(X), Y) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mor}(X, V(Y)).$$

For $X := V(Y)$ we get

$$\eta_{(V(Y),Y)} : \mathrm{Mor}(F(V(Y)), Y) \hookrightarrow\!\!\!\!\rightarrow \mathrm{Mor}(V(Y), V(Y)).$$

Thus

$$\psi_Y := \eta^{-1}_{(V(Y),Y)}(\mathrm{id}_{V(Y)}) : F(V(Y)) \to Y.$$

- It follows, that $\psi : F \circ V \to \mathrm{id}_{\mathfrak{C}}$ is a natural transformation, called the **counit** of (F, V, η).
- For every $Y \in \mathrm{Obj}(\mathfrak{C})$ the pair $(V(Y), \psi_Y)$ is a *universal arrow* from F to Y.

For an adjunction (F, V, ν) we get the adjunction $(V^{\mathrm{t}}, F^{\mathrm{t}}, \eta^{-1})$, and the unit of $(V^{\mathrm{t}}, F^{\mathrm{t}}, \eta^{-1})$ is the counit of (F, V, ν) (and vice versa).
(Using $F^{\mathrm{t}} : \mathfrak{S}^{\mathrm{t}} \to \mathfrak{C}^{\mathrm{t}}$ and $V^{\mathrm{t}} : \mathfrak{C}^{\mathrm{t}} \to \mathfrak{S}^{\mathrm{t}}$.)

Obtaining the adjunction-isomorphism from the (co)units

We have defined the adjoined pair (F, V) via the existence of the adjunction-isomorphism

$$\eta_{X,Y} : \mathrm{Mor}_{\mathfrak{C}}(F(X), Y) \hookrightarrow\!\!\!\!\!\rightarrow \mathrm{Mor}_{\mathfrak{S}}(X, V(Y))$$

which is natural in $X \in \mathrm{Obj}(\mathfrak{S})$ and $Y \in \mathrm{Obj}(\mathfrak{C})$. So we have mappings

$$f : F(X) \to Y \;\mapsto\; \eta_{X,Y}(f) : X \to V(Y)$$
$$g : X \to V(Y) \;\mapsto\; \eta^{-1}_{X,Y}(g) : F(X) \to Y.$$

Now these mappings are realised by the units $\varphi_X : X \to V(F(X))$ and the counits $\psi_Y : F(V(Y)) \to Y$:

$$\eta_{X,Y}(f) = V(f) \circ \varphi_X$$
$$\eta^{-1}_{X,Y}(g) = \psi_Y \circ F(g).$$

Adjoints via universal arrows to the functor

We have seen that given an adjunction (F, V, η), where $V : \mathfrak{C} \to \mathfrak{S}$, for every $X \in \mathfrak{S}$ (the "sets") we obtain a universal arrow $(F(X), \varphi_X)$ from X to V.

Consider the situation where we just have given the functor $V : \mathfrak{C} \to \mathfrak{S}$, but not F, and for every X we have a universal arrow $(F(X), \varphi_X)$ to V, where now we have just given $F : \mathrm{Obj}(\mathfrak{S}) \to \mathrm{Obj}(\mathfrak{C})$ and $\varphi : \mathrm{Obj}(\mathfrak{S}) \to \mathrm{Mor}(\mathfrak{S})$.

This data already *uniquely determines* a left adjoint F of V (that is, V has some left adjoint F', and V has exactly one left adjoint F which extends the given object map F and such that the unit of the adjunction is φ).

Adjoints via universal arrows from the functor

Dually, we have seen that given an adjunction (F, V, η), where $F : \mathfrak{S} \to \mathfrak{C}$, for every $X \in \mathfrak{C}$ we obtain a universal arrow $(V(X), \psi_X)$ from F to X.

Assume we have the functor $F : \mathfrak{S} \to \mathfrak{C}$, but not V, and for every X we have a universal arrow $(V(X), \psi_X)$ from F (where $V : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Obj}(\mathfrak{S})$ and $\psi : \mathrm{Obj}(\mathfrak{C}) \to \mathrm{Mor}(\mathfrak{C})$).

This data already *uniquely determines* a right adjoint V of F (that is, F has some right adjoint V', and F has exactly one right adjoint V which extends the given object map V and such that the counit of the adjunction is ψ).

Basic properties of adjunctions

1. Uniqueness of adjoints:
   - Two left adjoints of a functor are isomorphic.
   - Two right adjoints of a functor are isomorphic.
2. Preservation of limits and colimits:
   - Left adjoints preserve colimits.
   - Right adjoints preserve limits.

Some mnemonics

The uniqueness of adjoint functors is an extension of the fact, that universal arrows are unique up to isomorphism.

To remember which kind of adjoint (left? right??) respects limits or colimits, consider the category $\mathfrak{MON}$ of monoids and the adjoint pair (F, V), where V is the forgetful functor and F the “free” functor:

1. The set $\{1\}$ is terminal in $\mathfrak{SET}$, which maps via F to the monoid $\mathbb{N}_0$, which is not terminal in $\mathfrak{MON}$.
2. The monoid $\{0\}$ is initial in $\mathfrak{MON}$, but the set $\{0\}$ is not initial in $\mathfrak{SET}$.

Thus F does not respect limits (terminal objects are special products, which are special limits), while V does not respect colimits (initial objects are special coproducts, which are special colimits).

Properties of the unit and the counit

Consider an adjunction (F, V, η):

- Regarding F:
  - F is faithful iff the unit if pointwise monomorph.
  - F is full iff the unit is a pointwise retraction.
  - F is fully faithful iff the unit is an isomorphism.
- Regarding V:
  - V is faithful iff the counit if pointwise epimorph.
  - V is full iff the counit is a pointwise coretraction.
  - V is fully faithful iff the counit is an isomorphism.

So "typically" units are (pointwise) monomorph and counits are (pointwise) epimorph.

Normal examples

Recall for example the units and counits of the $\mathfrak{MON} \to \mathfrak{SET}$ situation:

- the units are the canonical injections $X \to V(F(X))$;
- the counits are the homomorphisms $F(V(M)) \to M$, which assign to every word over a monoid M the composition of its "letters".)

(Co)Limits and universal arrows

Consider a category $\mathfrak{J}$ (the "diagram") and a functor

$$D : \mathfrak{J} \to \mathfrak{C}, \quad \text{that is, } D \in \mathfrak{FUN}(\mathfrak{J}, \mathfrak{C}).$$

Recall that we have a canonical functor

$$\Delta : \mathfrak{C} \to \mathfrak{FUN}(\mathfrak{J}, \mathfrak{C})$$

which assigns to every object X the constant functor with value X. (Δ is an embedding except of the cases where $\mathfrak{J}$ is empty while $\mathfrak{C}$ has at least two objects).

- A **limit** of D is a universal object from Δ to D.
- A **colimit** of D is a universal object from D to Δ.

(As usual with natural transformations, we take the liberty here to strip off the attached domain- and codomain-categories in the triple-representation.)

Cones and cocones

Regarding limits of $D : \mathfrak{I} \to \mathfrak{C}$:

- The objects of $(\Delta \downarrow D)$, which are pairs (P, η), where η is a natural transformation from $\Delta(P)$ to D, are called **cones with base** D (the top of the cone is on the left, the base on the right; the direction is dictated by the direction of the arrows).
- Limits of D are **limit cones with base** D.

Regarding colimits of $D : \mathfrak{I} \to \mathfrak{C}$:

- The objects of $(D \downarrow \Delta)$, which are pairs (S, η), where η is a natural transformation from D to $\Delta(S)$, are called **cocones with base** D (the base of the cocone is on the left, the top on the right).
- Colimits of D are **colimit cocones with base** D.

Cones are also called **projective cones**, and cocones **inductive cones**.

Special limits

To specify J, we may just specify a general directed graph, and then the free category associated with this dgg is meant.

1. If J is discrete, then limits are called **products** and colimits are called **coproducts**.
2. If J is a doublet (the directed general graph consisting of the vertices $0, 1$ and a bunch of arrows from 0 to 1), then limits are called **equalisers**, and colimits are called **coequalisers**.
3. If J is
$$\begin{tikzcd} & \bullet \arrow[dl] \\ \bullet & \\ & \bullet \arrow[ul] \end{tikzcd}$$
, then limits are called **pullbacks**.
4. If J is
$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur] \arrow[dr] & \\ & \bullet \end{tikzcd}$$
, then colimits are called **pushouts**.

Complete and cocomplete categories

- A category $\mathfrak{C}$ is **complete** if it has all small limits, while it is **cocomplete** if it has all small colimits.
- $\mathfrak{C}$ is complete iff it has all small products and all binary equalisers, while $\mathfrak{C}$ is cocomplete iff it has all small coproducts and all binary coequalisers.
- For a given diagram category $\mathfrak{J}$, the existence of all limits resp. colimits in $\mathfrak{C}$ with source J is equivalent to $\Delta : \mathfrak{C} \to \mathfrak{FUN}(\mathfrak{J}, \mathfrak{C})$ having a right adjoint resp. a left adjoint.

A nice property of small categories, generalising that a quasi-order has all infimums iff it has all supremums:

A small category is complete iff it is cocomplete.

(In complete small categories *all* limits and colimits exist.)

Transportation of structures

Consider a functor $F : \mathfrak{C} \to \mathfrak{D}$.

We assume a notion of a certain "structure" $\mathcal{S}$ for $\mathfrak{C}$ as well as for $\mathfrak{D}$, involving the objects and morphisms as well as other (set-theoretical) objects; the structures in $\mathfrak{C}$ are denoted by $\mathcal{S}(\mathfrak{C})$, those in $\mathfrak{D}$ by $\mathcal{S}(\mathfrak{D})$.

A structure $S \in \mathcal{S}(\mathfrak{C})$ can be transported by F, obtaining $F(S)$, that is, applying F to all involved objects and morphisms, and leaving the rest intact. If $F(S) \in \mathcal{S}(\mathfrak{D})$, then we say that S is *transportable* via F.

Examples

Examples for structures transportable via all functors:

1. Single objects $X \mapsto F(X)$ and single morphisms $f \mapsto F(f)$.
2. Diagrams $\mathfrak{D} : J \to \mathfrak{C}$ are transported to $F \circ D : \mathfrak{J} \to \mathfrak{D}$.
3. Natural transformations $\eta : D_1 \to D_2$ for $D_1, D_2, : \mathfrak{J} \to \mathfrak{C}$ are transported to $F \circ \eta : F \circ D_1 \to F \circ D_2$ (a special case of the "horizontal" composition of natural transformations).
4. An object of the comma category $(D \downarrow \Delta_{\mathfrak{C}})$, where $\Delta_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{FUN}(\mathfrak{J}, \mathfrak{C})$ (in other words, a "cocone" for the diagram D), is transported to an "object" of $(F \circ D \downarrow \Delta_{\mathfrak{D}'})$, where $\mathfrak{D}'$ is the image of F. So here the transport creates some "damage", but this can be easily repaired by extending $\mathfrak{D}'$ to all of $\mathfrak{D}$.

Preservation and detection of properties

For a functor $F : \mathfrak{C} \to \mathfrak{D}$, a "structure notion" $\mathcal{S}$, and a property P of such structures:

1. F **preserves** (or **respects**) P for $\mathcal{S}$ if for each $S \in \mathcal{S}(\mathfrak{C})$ transportable via F we have, that if S has property P, then also $F(S)$ has property P.
2. F **detects** (or **reflects**) P for $\mathcal{S}$ if for each $S \in \mathcal{S}(\mathfrak{C})$ transportable via F we have, that if $F(S)$ has property P then also S has property P.

These two properties are relatively simple, because they focus on a structure S in $\mathfrak{C}$. After some examples we then consider the case that we want to "transport back" some structure in $\mathfrak{D}$.

Basis examples

- Each functor F respects retractions, coretractions and isomorphisms.
- Each faithful F detects monomorphisms and epimorphisms.
- F is called **(co)continuous** if F respects (co)limits.
- F is (co)continuous iff F respects (co)products and (co)equalisers.
- If F
  - respects pullbacks, then F respects monomorphisms
  - respects pushouts, then F respects epimorphisms.

  (Note that a morphism f is mono (epi) iff id is a pullback (pushout) of (f, f).)
- Thus left adjoints respect epimorphisms, and right adjoints respect monomorphisms.

A note on representable functors

A functor $F : \mathfrak{C} \to \mathfrak{SET}$ is called **representable** if there is some object $A \in \mathrm{Obj}(\mathfrak{C})$ such that F is isomorphic to the (covariant) hom-functor

$$\mathrm{Mor}_{\mathfrak{C}}(A, -) : \mathfrak{C} \to \mathfrak{SET},\ X \in \mathrm{Obj}(\mathfrak{C}) \mapsto \mathrm{Mor}(A, X).$$

A contravariant functor $F : \mathfrak{C}^{\mathrm{t}} \to \mathfrak{SET}$ is called **representable** if there is some object $A \in \mathrm{Obj}(\mathfrak{C})$ such that F is isomorphic to the contravariant hom-functor

$$\mathrm{Mor}_{\mathfrak{C}}(-, A) : \mathfrak{C}^{\mathrm{t}} \to \mathfrak{SET},\ X \in \mathrm{Obj}(\mathfrak{C}) \mapsto \mathrm{Mor}(X, A).$$

- F is representable iff there is a universal arrow from $\{0\}$ to F.
- Every representable functor is continuous.
- Representable contravariant functors on the other hand transform colimits into limits.

Lifting and creation of limits

For a functor $F : \mathfrak{C} \to \mathfrak{D}$ and a diagram $D : \mathfrak{J} \to \mathfrak{C}$:

1. F **lifts limits** for D if for a limit P of $F \circ D$ there is a limit P_0 of D which is transported via F to P.
2. F **uniquely lifts limits** for D if for a limit P of $F \circ D$ there is a unique limit P_0 of D transported via F to P.
3. F **creates limits** for D if if for a limit P of $F \circ D$ there is a unique cone P_0 with base D transported via F to P, and this unique cone is a limit cone for D.

If no diagram is specified, then the lifting- or creation-property holds (at all) if it holds for all $D : \mathfrak{J} \to \mathfrak{C}$ with small $\mathcal{J}$.

F creates limits iff
F detects limits and F uniquely lifts limits.

For colimits we have the corresponding dual notions.

Basis usage of lifts

Consider a functor $F : \mathfrak{C} \to \mathfrak{D}$:

- If F lifts limits, and $\mathfrak{D}$ is complete, then $\mathcal{C}$ is complete.
- If F lifts colimits, and $\mathfrak{D}$ is cocomplete, then $\mathfrak{C}$ is cocomplete.

It seems that there are no general criterions for lifting (co)limits, but that lifts are given for special (general) situations. For us the main example is, using categories $\mathfrak{C}, \mathfrak{S}$, objects $X \in \mathrm{Obj}(\mathfrak{S})$ and a functor $V : \mathfrak{C} \to \mathfrak{S}$:

1. V continuous $\Rightarrow P : (X \downarrow V) \to \mathfrak{C}$ creates limits.
2. V cocontinuous $\Rightarrow P : (V \downarrow X) \to \mathfrak{C}$ creates colimits.

Thus:

- If V is continuous and $\mathfrak{C}$ complete, then all comma categories $(X \downarrow V)$ are complete.
- If V is cocontinuous and $\mathfrak{C}$ cocomplete, then all comma categories $(V \downarrow X)$ are cocomplete.

Regarding full embeddings

Of course, isomorphisms between categories respect and create limits. Close to isomorphisms are full embeddings.

So consider a full embedding $F : \mathfrak{C} \to \mathfrak{D}$ (that is, F is an isomorphism of $\mathfrak{C}$ to a full subcategory $F(\mathfrak{C})$ of $\mathfrak{D}$; note however that the codomain of F is $\mathfrak{D}$).

- F detects limits and colimits.
- F respects (co)limits iff every (co)limit in $F(\mathfrak{C})$ is also a (co)limit in $\mathfrak{D}$.
- If F lifts (co)limits, then the lifting is unique, and F actually creates (co)limits.
- Now assume that F respects (co)limits and that $\mathfrak{C}$ is (co)complete:

  F lifts (co)limits iff $F(\mathfrak{C})$ is isomorphism-closed.

  (To show the direction from left to right, consider (co)products of one-object-families.)

Weakening initiality and terminality

Consider a category $\mathfrak{C}$. Call an object $A \in \mathrm{Obj}(\mathfrak{C})$

- a **source** if for all $Y \in \mathrm{Obj}(\mathfrak{C})$ there is $f : A \to Y$;
- a **sink** if for all $X \in \mathrm{Obj}(\mathfrak{C})$ there is $f : X \to A$.

More generally, call $\mathbb{A} \subseteq \mathrm{Obj}(\mathfrak{C})$ (where $\mathbb{A}$ is small(!))

- a **set-source** if for all $Y \in \mathrm{Obj}(\mathfrak{C})$ there is $A \in \mathbb{A}$ and $f : A \to Y$;
- a **set-sink** if for all $X \in \mathrm{Obj}(\mathfrak{C})$ there is $A \in \mathbb{A}$ and $f : X \to A$.

Obviously:

1. A category having a source does not need to have an initial object.
2. A category having a sink does not need to have a terminal object.

Solution set condition for initiality/terminality

Lemma *A complete category with some set-source has an initial object.*

- The dual version is: A cocomplete category with some set-sink has a terminal object.
- This lemma is the germ of what could be considered the underlying theme of this talk (and of the adjoint functor theorem): "Connecting the two sides", i.e., switching from the "unnamed side" of category theory to the "co-side", or, more specifically, switching from the limits to the colimits.

Let us start the proof; let $\mathfrak{C}$ be a complete category and $\mathbb{A}$ some set-source. The first strike is easy: $S := \prod \mathbb{A}$ is a source of $\mathfrak{C}$. In the remainder of the proof only S will be used.

Killing endomorphisms

In general S need not to be an initial object.

In order to be initial, S must not have non-trivial endomorphisms. So consider an equaliser

$$\sigma : E \to S$$

for the set of endomorphisms of S. We want to show that E now is an initial object. So consider some object X and two morphisms $f, g : E \to X$. We need to show that $f = g$ holds.

Let $\alpha : E' \to E$ be an equaliser of f, g. We need to show that α is an isomorphism. Since every equaliser is a monomorphism, this is equivalent to showing that α is a retraction.

Round the circle

Since S is a source, we get some morphism $\pi : S \to E'$, yielding the diagram

$$\begin{tikzcd} & E' \arrow[d, "\alpha"'] & \\ & E \arrow[r, "\sigma"'] \arrow[dl, "f"] \arrow[dr, "g"'] & S \arrow[ul, "\pi"'] \\ X & & X \end{tikzcd}$$

For the two endomorphisms $\sigma \circ \alpha \circ \pi$ and id_S of S:

$$(\sigma \circ \alpha \circ \pi) \circ \sigma = \mathrm{id}_S \circ \sigma, \Rightarrow \sigma \circ (\alpha \circ \pi \circ \sigma) = \sigma \circ \mathrm{id}_E \,.$$

σ is a monomorphism, so $\alpha \circ \pi \circ \sigma = \mathrm{id}_E$. QED

The adjoint functor theorem again

Theorem *Consider a complete category* $\mathfrak{C}$ *with small morphism-sets, and a functor* $V : \mathfrak{C} \to \mathfrak{S}$*.*

Then V *has a left adjoint if and only if* V *is "continuous, and for every every* $X \in \mathrm{Obj}(\mathfrak{S})$ *we have a "solution set", that is, a set-source in* $(X \downarrow V)$*; explicitely stated, we require a family* $(A_i)_{i \in I}$ *of objects in* $\mathfrak{C}$ *together with a family* $(f_i)_{i \in I}$ *of morphisms* $f_i : X \to V(A_i)$ *in* $\mathfrak{S}$ *such that*

for all $A \in \mathrm{Obj}(\mathfrak{C})$ *and all* $f : X \to V(A)$ *there is some* $i \in I$ *together with some morphism* $g : A_i \to A$ *in* $\mathfrak{C}$ *with*
$$f = V(g) \circ f_i.$$

Finally the proof

Consider $X \in \mathrm{Obj}(\mathfrak{C})$.

1. We have to show the existence of a universal arrow from X to V.
2. In other words, we have to show the existence of an initial object in $(X \downarrow V)$.
3. Since V is continuous and $\mathfrak{C}$ is complete, also $(X \downarrow V)$ is complete.
4. Now the solution set condition just means that $(X \downarrow V)$ has a set-source.
5. Thus by the previous lemma we get the initial object in $(X \downarrow V)$.

QED

Generators and cogenerators

Consider a category $\mathfrak{C}$. A set $\mathbb{A}$ of objects in $\mathfrak{C}$ is a

- **generator set**, if for every pair $f, g : X \to Y$ of morphisms in $\mathfrak{C}$ we have that $f \neq g$ if and only if there is some $A \in \mathbb{A}$ and a morphism $\alpha : A \to X$ with $f \circ \alpha \neq g \circ \alpha$.
- **cogenerator set**, if for every pair $f, g : X \to Y$ of morphisms in $\mathfrak{C}$ we have that $f \neq g$ if and only if there is some $A \in \mathbb{A}$ and a morphism $\alpha : Y \to A_i$ with $\alpha \circ f \neq \alpha \circ g$.

If $|\mathbb{A}| = 1$, then we just speak of a **generator** resp. a **cogenerator**. For example, in the category $\mathfrak{SET}$ of sets:

1. $\{0\}$ is a generator.
2. $\{0, 1\}$ is a cogenerator.

Eliminating the solution set condition

Lemma *Consider a complete category $\mathfrak{C}$ with small morphism-sets, and a functor $V : \mathfrak{C} \to \mathfrak{S}$. Furthermore assume that*

1. *$\mathfrak{C}$ has a cogenerator set.*
2. *$\mathfrak{C}$ is concrete.*
3. *Every monomorphism in $\mathfrak{C}$ is injective.*
4. *Every bijective morphism in $\mathfrak{C}$ is an isomorphism.*

Then V has a left adjoint if and only if V is continuous.

Left adjoints for embeddings

We consider now the problem of finding a left adjoint for some embedding into a category $\mathfrak{C}$.

W.l.o.g. we can take this embedding as the canonical embedding of a sub-category $\mathfrak{U}$ of $\mathfrak{C}$.

A left adjoint $F : \mathfrak{C} \to \mathfrak{U}$ of this embedding now is called a **reflector** for $\mathfrak{U}$, and is characterised by canonical isomorphisms for

$$\mathrm{Mor}_{\mathfrak{U}}(F(X), Y) \cong \mathrm{Mor}_{\mathfrak{C}}(X, Y)$$

for $X \in \mathrm{Obj}(\mathfrak{C})$ and $Y \in \mathrm{Obj}(\mathfrak{U})$.

Reflective and coreflective subcategories

- A reflector for a subcategory $\mathfrak{U}$ of some category $\mathfrak{C}$ is a left-adjoint of the canonical embedding, and if such a reflector exists, then $\mathfrak{U}$ is called a **reflective subcategory** of $\mathfrak{C}$.
- Dually, a coreflector for a subcategory $\mathfrak{U}$ of some category $\mathfrak{C}$ is a right-adjoint of the canonical embedding, and if such a coreflector exists, then $\mathfrak{U}$ is called a **coreflective subcategory** of $\mathfrak{C}$.

Left adjoints for full embeddings

Since every embedding is faithful, the counits of the adjunction are always epimorphisms.

Now in the important situation, that $\mathfrak{U}$ is a *full subcategory*, the counits are isomorphisms, and then w.l.o.g. the counits can be assumed to be the identities and F is identical on $\mathfrak{U}$.

So in this situation the inverse of the adjunction-isomorphism takes a very simple form:

$$g : X \to Y \;\; \hookrightarrow\!\!\!\!\rightarrow \;\; F(g) : F(X) \to Y.$$

The other direction is given by

$$f : F(X) \to Y \;\; \hookrightarrow\!\!\!\!\rightarrow \;\; f \circ \varphi_X : X \to Y,$$

using the unit $\varphi_X : X \to F(X)$ of the adjunction.

Meaning of full (co)reflective subcategories

Assume that $\mathfrak{U}$ is a full subcategory of $\mathfrak{C}$.

1. $\mathfrak{U}$ is reflective iff there exists a functor $F : \mathfrak{C} \to \mathfrak{U}$, identical on $\mathfrak{U}$, such that for every *general object* $X \in \mathrm{Obj}(\mathfrak{C})$ and for every *special object* $S \in \mathrm{Obj}(\mathfrak{U})$ the morphisms from X to S are, via F, in 1-1 correspondence to the morphisms from $F(X)$ to S — $F(X)$ is the *most general specialisation* of X loosing nothing w.r.t. morphisms **on** X into special spaces.
2. $\mathfrak{U}$ is coreflective iff there exists a functor $G : \mathfrak{C} \to \mathfrak{U}$, identical on $\mathfrak{U}$, such that for every *general object* $Y \in \mathrm{Obj}(\mathfrak{C})$ and for every *special object* $S \in \mathrm{Obj}(\mathfrak{U})$ the morphisms from S to Y are, via G, in 1-1 correspondence to the morphisms from S to $G(Y)$ — so $G(Y)$ is the *most general specialisation* loosing nothing w.r.t. morphisms **into** Y from special spaces.

Inheriting (co)limits

Consider a (co)complete category $\mathfrak{C}$ and a subcategory $\mathfrak{U}$. We want to construct a (co)reflection for $\mathfrak{U}$, that is a left resp. right adjoint for $J : \mathfrak{U} \hookrightarrow \mathfrak{C}$. To apply the adjoint functor theorem, we need $\mathfrak{U}$ to be (co)complete, and J to respect (co)limits.

The easiest way here is to inherit all (co)limits from $\mathfrak{C}$ (if possible):

$\mathfrak{U}$ **inherits (co)limits** from $\mathfrak{C}$ if every limit in $\mathfrak{C}$ for objects and morphisms in $\mathfrak{U}$ can be realised in $\mathfrak{U}$.

If $\mathfrak{U}$ inherits (co)limits, then $\mathfrak{U}$ is (co)complete and J is (co)continuous.

Constructing reflections

Lemma *Consider a concrete and complete category $\mathfrak{C}$.*

Let $\mathfrak{U}$ be a subcategory such that

1. *$\mathfrak{U}$ inherits limits from $\mathfrak{C}$.*
2. *Every monomorphism in $\mathfrak{U}$ is injective.*
3. *Every bijective morphism in $\mathfrak{U}$ is an isomorphism.*
4. *$\mathfrak{U}$ has a cogenerator set.*

Then $\mathfrak{U}$ is a reflective subcategory of $\mathfrak{C}$.

Direct examples for reflections I

- The category of all preordered (or thin) categories is a reflective full i-c subcategory of $\mathfrak{CAT}$.
- The category $\mathfrak{POR D}$ of all partially ordered sets is a reflective full i-c subcategory of $\mathfrak{QORD}$.

Direct examples for reflections II

- The category of all T_0-spaces is a reflective full i-c subcategory of $\mathfrak{TOP}$.
- $\mathfrak{MON}$ can be canonically identified with the (non-full) subcategory of $\mathfrak{SGR}$ given by all semigroups with neutral element (then uniquely determined) together with neutral-element preserving homomorphisms. Then $\mathfrak{MON}$ is a reflective sub-category of $\mathfrak{SGR}$. (Note that here the reflector is not identical on the monoids, and the counit is only pointwise epimorph; the reflector is faithful (actually an embedding), and accordingly all units are pointwise monomorph.)

Basic (direct) examples for coreflections

- The category $\mathfrak{SET}$ of all sets has a canonical i-c full embedding into $\mathfrak{PORD}$ (using discrete orders). Then $\mathfrak{SET}$ is a coreflective subcategory of $\mathfrak{PORD}$.
- The category of all discrete categories is a coreflective (i-c full) subcategory of $\mathfrak{CAT}$.
- $\mathfrak{QORD}$ is canonically isomorphic to $\mathfrak{STOP}$, the i-c full subcategory of $\mathfrak{TOP}$ given by the condition, that arbitrary intersections of open sets are open. $\mathfrak{STOP}$ is a coreflective subcategory of $\mathfrak{TOP}$.

Reminder

- A topological space is a pair (X, τ), where X is a set and $\tau \subseteq \mathbb{P}(X)$ is closed under finite intersections and arbitrary unions.
- The category $\mathfrak{TOP}$ of topological spaces has a natural i-c full embedding (via dualisation) into the category of all closure systems (pairs $(X, \mathbb{M})$, where $\mathbb{M} \subseteq \mathbb{P}(X)$ is closed under arbitrary intersections), which in turn is an i-c full subcategory of $S(P^-)$.
- The category $\mathfrak{HTOP}$ of T_2-topological spaces (two distinct points can be separated by disjoint open sets) is an i-c full subcategory of $\mathfrak{TOP}$.
- A topological space is quasi-compact if every open cover has a finite sub-cover. A topological space X is compact if X is T_2 and quasi-compact.
- The category $\mathfrak{COMP}$ of compact topological spaces is an i-c full subcategory of $\mathfrak{TOP}$.

The forgetful functor for all spaces

Consider $V : \mathfrak{TOP} \to \mathfrak{SET}$.

- $D : \mathfrak{SET} \to \mathfrak{TOP}$, $S \mapsto (S, \mathbb{P}(X))$ is the canonical left adjoint of V.

  (D is also an i-c full embedding, and a right-inverse of V.)
- $I : \mathfrak{SET} \to \mathfrak{TOP}$, $S \mapsto (S, \{\emptyset, S\})$ is the canonical right adjoint of V.

  (I is also an i-c full embedding, and a right-inverse of V.)

Thus V respects all limits and colimits.

- However, V for example does neither detect products nor coproducts.
- On the other hand, by considering initial and final structures, we see that V uniquely lifts limits and colimits.

So $\mathfrak{TOP}$ is complete and cocomplete.

The subcategories of (in)discrete spaces

We have seen the adjoint pairs (D, V) and (V, I), Regarding the images

- $D(\mathfrak{SET})$, the i-c full subcategory of discrete spaces (every subset is open)
- $I(\mathfrak{SET})$, the i-c full subcategory of indiscrete spaces (no subset other than the empty set and the full set is open)

we can consider the functors

- $V_D : \mathfrak{TOP} \to D(\mathfrak{SET})$
- $V_I : \mathfrak{TOP} \to I(\mathfrak{SET})$.

We obtain that

- $D(\mathfrak{SET})$ is a bi-coreflective subcategory of $\mathfrak{TOP}$
- $I(\mathfrak{SET})$ is a bi-reflective subcategory of $\mathfrak{TOP}$.

Remarks regarding monos and epis

Since $\mathfrak{TOP}$ is a concrete category, every monomorphism is injective, every epimorphism is surjective (and this is inherited for every sub-category). By the existence of the left- and right-adjoint we also get the inverse. However, for that we do not need such strong means:

- For every sub-category of $\mathfrak{TOP}$ including all constant maps (for example for all full sub-categories) the monomorphisms are exactly the injective maps.
- For every sub-category of $\mathfrak{TOP}$ including some indiscrete space with at least two objects the epimorphisms are exactly the surjective maps.

We see a (typical) asymmetry: For many interesting sub-categories, monomorphisms are the injective maps, but often non-trivial indiscrete spaces are excluded (because of separation concerns), and so whether every epimorphisms needs to be surjective is open there.

Regarding limits

To construct the limits in $\mathfrak{TOP}$, it suffices to know products and equalisers:

- The (canonical) product of a family $(X_i)_{i \in I}$ of spaces is the smallest topology on the product set $\prod_{i \in I} V(X_i)$, such that all projections are continuous (in other words, the initial topology for the projection maps).
- The (canonical) equaliser of a family $f_i : X \to Y$ of continuous maps is the sub-space given by $\{x \in X \mid (i \in I \mapsto f_i(x)) \text{ constant}\}$ together with the canonical embedding.

Compact spaces and their category

Regarding the category $\mathfrak{COMP}$ and the forgetful functor $V : \mathfrak{COMP} \to \mathfrak{SET}$:

- $\mathfrak{COMP}$ is an i-c full subcategory of $\mathfrak{TOP}$.
- $\mathfrak{COMP}$ is **balanced**:
  - The monomorphisms are exactly the injective maps (which are now closed maps, whence topological embeddings).
  - The epimorphisms are exactly the surjective maps.
- $\mathfrak{COMP}$ is closed under products in $\mathfrak{TOP}$ (Tychonoff's theorem).
- $\mathfrak{COMP}$ is closed under equaliser formation in $\mathfrak{TOP}$ (equalising sets are closed in Hausdorff spaces).

Thus $\mathfrak{COMP}$ inherits limits, whence V respects limits. Using balancedness appropriately, we also get

V respects and creates limits.

Applying the solution set condition

Our aim is to get a left-adjoint for $V : \mathfrak{COMP} \to \mathfrak{SET}$. Applying the solution set condition, for a set S we need to show

- the existence of a family $(A_i)_{i \in I}$ of compact spaces together with maps $f_i : S \to V(A_i)$,
- such that for all compact topological spaces A and every map $f : S \to V(A)$ there is some $i \in I$ and a continuous map $g : A_i \to A$ such that $f = V(G) \circ f_i$.

To construct such families $(A_i)_{i \in I}$ is left as an exercise (we want to apply stronger means).

$\mathfrak{COMP}$ as a reflective subcategory of $\mathfrak{TOP}$

We know that $\mathfrak{TOP}$ is concrete and complete.

1. $\mathfrak{COMP}$ inherits limits.
2. The monomorphisms in $\mathfrak{COMP}$ are exactly the injective continuous maps.
3. The isomorphisms in $\mathfrak{COMP}$ are exactly the bijective continuous maps.

To apply the lemma about the existence of reflections, we need a cogenerator (set). Consider the unit interval $I = [0, 1]$ (a compact space):

> By Urysohn's lemma for every compact space K and $x, y \in K$, $x \neq y$, there exists a continuous $f : K \to I$ with $f(x) = 0$ and $f(y) = 1$.

Thus I is a generator for $\mathfrak{COMP}$. Thus $\mathfrak{COMP}$ is a reflective subcategory of $\mathfrak{TOP}$.

The Čech-Stone compactification

By the lemma about the existence of reflections, we now get:

There exists a reflection $\beta' : \mathfrak{TOP} \to \mathfrak{COMP}$.

$\beta'(X)$ for a topological space X is called the **Čech-Stone compactification** of X:

- For X completely regular $\beta'(X)$ is a compactification of X in the normal sense (while otherwise no compactifications in the normal sense exists).
- Restricting β' to $\mathfrak{HTOP}$ we obtain an epi-reflector.
- Restricting β' further to $D(\mathfrak{SET})$, via the identification of discrete spaces and sets we obtain a left adjoint of the forgetful functor $\mathfrak{COMP} \to \mathfrak{SET}$.

In the remainder, a concrete construction for $\beta(X)$ for discrete spaces X is outlined.

The Čech-Stone compactification of discrete spaces

Given a discrete topological space X, the topology of βX is as follows:

1. For $A \subseteq X$ let
$$[A] := \{F \in \beta X : A \in F\}$$
and let $\mathbb{B} := \{[A] : A \in \mathbb{P}(X)\}$.
2. $\mathbb{B}$ is stable under finite intersection, and thus the closure of $\mathbb{B}$ creates a topology on βX.

The basic facts about the topological space βX:

- $\beta(X)$ is compact.
- X is considered to be a subspace of βX via the (topological) embedding $x \in X \mapsto \mathfrak{G}\mathfrak{F}(x) \in \beta X$.
- X is a discrete and dense subset of βX.

The left adjoint of $V : \mathfrak{COMP} \to \mathfrak{SET}$ again

Now the map $X \in \mathrm{Obj}(\mathfrak{SET}) \mapsto \beta X \in \mathfrak{COMP}$ yields a functor (since there are no restrictions on maps on discrete spaces) $\beta : \mathfrak{SET} \to \mathfrak{COMP}$:

$$(\beta, V) \text{ is an adjoint pair.}$$

(And thus, due to the uniqueness of adjoints, β is isomorphic to β' restricted to discrete spaces.)

$\mathfrak{End}$