1. Simplify
\[ \log_a(a^x \cdot a^z) \] (1)
\[ \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor \] (2)
\[ \sum_{i=1}^{n} (i^2 - (i - 1)^2). \] (3)

Give some justifications. \([15 \text{ marks}]\)

2. Give \(\Theta\)-expressions which are as simple as possible:
\[ 5n^3 - 6n^2 + \sqrt{\sin(n)} = \Theta(?) \] (4)
\[ 2^n + n^{1000} = \Theta(?) \] (5)
\[ 2^n + 3^n = \Theta(?) \] (6)

Give some justifications. \([15 \text{ marks}]\)

3. Sort the expressions in ascending order of growth (i.e., if \(f(x)\) is left of \(g(x)\), then you need to have \(f(x) = O(g(x))\):
\[ \sqrt{n}, 2^{(n^2)}, 2^{\log n}, 2^{\log_{10} n}, n^3, n^2, e^n, 2^n, n!, n^n. \]

\([20 \text{ marks}]\)

4. Solve the following recurrences by \(\Theta\)-expressions:
\[ T(n) = 3T\left(\frac{n}{2}\right) + 5 \] (7)
\[ T(n) = 16T\left(\frac{n}{4}\right) + n^2 \] (8)
\[ T(n) = T\left(\frac{n}{2}\right) + \sqrt{n}. \] (9)

Show the details of your computations. \([15 \text{ marks}]\)

5. Show how to sort 4 arbitrary numbers using (only) 5 comparisons. Argue that your method always works. \([20 \text{ marks}]\)

6. Consider an array \(A[1\ldots n]\) of \(n\) distinct numbers. If \(i < j\) and \(A[i] > A[j]\), then the pair \((i, j)\) is called an inversion of \(A\). For example \((3, 2, 1)\) has three inversions.

(a) List the five inversions of the array \((2, 3, 8, 6, 1)\). \([5 \text{ marks}]\)

(b) What array with elements from the set \(\{1, \ldots, n\}\) has the most inversions? How many does it have? \([10 \text{ marks}]\)

(c) What is the relationship between the running time of insertion sort and the number of inversions in the input array? Justify your answer. \([10 \text{ marks}]\)

(d) Give an algorithm that determines the number of inversions in any permutation on \(n\) elements in \(O(n \log n)\) worst-case time. Argue that it’s correct and has the specified time-bound. (Hint: Modify merge sort.) \([20 \text{ marks}]\)