Effectivity of Regular Spaces

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Abstract. General methods of investigating effectivity on regular Hausdorff (T_3) spaces is considered. It is shown that there exists a functor from a category of T_3 spaces into a category of domain representations. Using this functor one may look at the subcategory of effective domain representations to get an effectivity theory for T_3 spaces. However, this approach seems to be beset by some problems. Instead, a new approach to introducing effectivity to T_3 spaces is given. The construction uses effective retractions on effective Scott–Ershov domains. The benefit of the approach is that the numbering of the basis and the numbering of the elements are derived at once.

1 Introduction

Domain theory has been used as a successful means to study effectivity on various spaces via domain representations of the spaces. This is due to the natural effectivity theory for domains and the inherit notion of approximation that exists within domains.

Representations of topological spaces by domains or embeddings of topological spaces into domains have been studied by several people. Weihrauch and Schreiber [20] considered embeddings of metric spaces into cpos with weight and distance. Stoltenberg-Hansen and Tucker [16, 17] introduced the notion of domain representability. Edalat [4–7] has used embeddings into continuous dcpos to study integration, measures and fractals. Edalat and Heckmann [8] and di Gianantonio [3] among others have also studied similar notions. Ershov's [9] representation of the Kleene–Kreisel continuous functionals is an early example of a domain representation.

A result by the author [2] characterises the T_3 spaces as exactly the ones that have a certain type of domain representations. It is therefore a natural desire to study the effectivity theory induced by this class of domain representations. This is the main aim of the paper.

Investigations of effectivity on topological spaces, even much weaker topologies, has been studied by Spreen [14] and Kreitz and Weihrauch [11, 19] among others.

Section 2 gives some basic definitions and recalls some results. Section 3 gives a lifting result for continuous functions to countable based domain representations. Section 4 gives a full and faithful functor from a category of topological

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spaces into a category of domain representations. The problem here is that there is no canonical choice of a numbering for the basis. Section 5 is the main section introducing the new approach to constructing domain representations and in particular effective domain representations. The effective representations that are considered are those that can be obtained from effective retractions on effective domains. The primary benefit of this approach is that it gives numberings for both the elements and for a basis for the topology. Although it is shown that any space has representations constructed according to this approach, it is not clear that every interesting effective representation can be obtained in this manner. For the effective representations of T_3 spaces obtained some effective properties are investigated. For example, it is not always possible to compute the operation that takes an index of a filter base of a point to an index of the point.

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2 Preliminaries

A space is T_3 if it is regular and T_1. Thus, any T_3 space is Hausdorff. The topological closure and interior of a set S is denoted by $\overline{S}$ and S° respectively.

We will use $\langle \cdot, \cdot \rangle$ to denote some standard recursive pairing operation on the natural numbers ω. The recursive projections π_0 and π_1 are assumed to satisfy $\pi_0\langle m, n\rangle = m$ and $\pi_1\langle m, n\rangle = n$. Fix $(W_n)_{n\in\omega}$ to be a standard enumeration of the r.e. sets.

We assume familiarity with the notion of domain representations [2, 17] and of domain theory, in particular, the theory of effective domains [15]. By a domain in this paper is meant a Scott–Ershov domain, i.e., a consistently complete algebraic cpo. We denote the compact elements of a domain D by D_{c}.

Definition 2.1. A *domain representation* of a topological space X is a tuple $(D, D^{\mathrm{R}}, \rho)$ such that D is a domain, D^{R} is a subset of D, and $\rho\colon D^{\mathrm{R}} \to X$ is a quotient map.

A domain representation is *upwards-closed* if D^{R} is an upper subset of D, and if $x \sqsubseteq y$ and $x \in D^{\mathrm{R}}$ implies $y \in D^{\mathrm{R}}$ and $\rho(x) = \rho(y)$.

If $(D, D^{\mathrm{R}}, \rho)$ is a domain representation of X and there exists an topological embedding $\eta\colon X \to D^{\mathrm{R}}$ then the tuple $(D, D^{\mathrm{R}}, \rho, \eta)$ is a *retract* domain representation of X.

We will usually consider only upwards-closed retract domain representation. The following two theorems are proven in [2].

Theorem 2.2. *Any T_3 space has a dense upwards-closed retract domain representation.*

In fact, the proven result is stronger in that the obtained domain representation has further nice properties. These extra properties will not be used in this paper, however.

Theorem 2.3. *Let $(D, D^{\mathrm{R}}, \rho, \eta)$ be an upwards-closed retract representation of X. Then X is T_3.*

The following easy result shows that functions induced from domain representations are continuous.

Proposition 2.4. *Let $(D, D^{\mathrm{R}}, \varphi)$ and $(E, E^{\mathrm{R}}, \psi)$ be domain representations of X and Y respectively. Let $\bar{f}: D \to E$ be continuous such that $\bar{f}[D^{\mathrm{R}}] \subseteq E^{\mathrm{R}}$ and assume $\bar{f}$ respects the equivalence relations induced by φ and ψ. Then $\bar{f}$ induces a unique continuous function $f: X \to Y$.*

3 Liftings of functions to non-dense representations

It is known that a continuous functions defined on a dense subset of a domain D into a Scott–Ershov domain can be extended to the domain D. See for example [10]. Following an idea of Geir Waagbø [18], we show that denseness is not needed for countably based coherent domains. The actual proof is a generalisation of the proof given by Dag Normann [12].

Definition 3.1. A domain representation is *coherent* if the domain is coherent, i.e., if any inconsistent finite set of elements contains an inconsistent pair of elements.

Examples of coherent domain representations include the flat domain representation of any countable set, and the usual interval representation of the reals.

Lemma 3.2. *Let D be a domain and E a coherent domain, then the function space $[D \to E]$ is coherent.*

Proof. Let $F = \{f_1, \ldots, f_n\}$ be a finite pairwise consistent set of functions in $[D \to E]$. Define a function $g: D \to E$ by

$$g(x) = \bigsqcup_{1 \le i \le n} f_i(x)\,.$$

Since F is pairwise consistent, it follows that $\{f_i(x) : 1 \le i \le n\}$ is a pairwise consistent set in the coherent domain E and therefore consistent. Hence $g(x)$ is well-defined for all $x \in D$. Clearly, $f_i \sqsubseteq g$ for all $i = 1, \ldots, n$. Let $h: D \to E$ be an upper bound of F. For all $i = 1, \ldots, n$ and all $x \in D$ we have $f_i(x) \sqsubseteq h(x)$, and hence $g(x) = \bigsqcup_{1 \le i \le n} f_i(x) \sqsubseteq h(x)$. Thus $g \sqsubseteq h$, i.e., g is the supremum $\bigsqcup_{1 \le i \le n} f_i$.

It remains to show that g is continuous. Assume that $x \sqsubseteq y$. By monotonicity of the function f_i it follows that

$$g(x) = \bigsqcup_{1 \le i \le n} f_i(x) \sqsubseteq \bigsqcup_{1 \le i \le n} f_i(y) = g(y)\,.$$

Assume that A is a directed subset of D and let $x = \bigsqcup_D A$. Then

$$g(x) = \bigsqcup_{1 \le i \le n} f_i(\bigsqcup_D A) = \bigsqcup_{1 \le i \le n} \bigsqcup_E f_i[A] = \bigsqcup_E \bigsqcup_{1 \le i \le n} f_i[A] = \bigsqcup_E g[A] .$$

□

Let (D, D^{R}, ν) be a countably based domain representation of X and $(E, E^{\mathrm{R}}, \rho, \eta)$ be a coherent retract domain representation of Y. Let $\varphi: X \to Y$ be a continuous function. We will lift φ to a domain function $g: D \to E$.

Define $f: D^{\mathrm{R}} \to E^{\mathrm{R}}$ by $f = \eta\varphi\nu$. We will show that there exists a function $g: D \to E$ such that $f = g|_{D^{\mathrm{R}}}$. Let $(a_n, b_n)_{n\in\omega}$ be an enumeration of all pairs (a_n, b_n) such that

(i) $\uparrow a_n \cap D^{\mathrm{R}} \neq \emptyset$, and
(ii) $f[\uparrow a_n \cap D^{\mathrm{R}}] \subseteq \uparrow b_n$.

Let Γ be the set of all basic step functions $\langle a; b\rangle$ for which there exists an n such that

(i) $a_n \sqsubseteq a$ and $b \sqsubseteq b_n$, and
(ii) for all $i < n$, b_i and b_n inconsistent implies that a and a_i inconsistent.

Lemma 3.3. *Any finite subset of Γ is consistent.*

Proof. By coherency it is sufficient to show that any $\langle a; b\rangle$ and $\langle a'; b'\rangle$ in Γ are consistent. There exists n and n' witnessing that the step functions belong to Γ. If $n = n'$ then $b \sqsubseteq b_n$ and $b' \sqsubseteq b_{n'} = b_n$ showing that the step functions are consistent. Assume that $n < n'$, the remaining case is symmetric. Suppose b and b' are inconsistent. Since $b \sqsubseteq b_n$ and $b' \sqsubseteq b_{n'}$ we have that b_n and $b_{n'}$ are inconsistent. By condition (ii) in the definition of Γ it follows that a_n and a' are inconsistent. Hence, since $a_n \sqsubseteq a$, we have that a and a' are inconsistent. Thus, the step functions are consistent. □

Define g to be the function obtained from the ideal generated by Γ.

Lemma 3.4. *The function g is an extension of f.*

Proof. Clearly, $g|_{D^{\mathrm{R}}} \sqsubseteq f$. Let $x \in D^{\mathrm{R}}$, and let $b \sqsubseteq fx$. By continuity, the preimage $f^{-1}[\uparrow b]$ is open. Thus, there exists $a \sqsubseteq x$ such that $\uparrow a \subseteq f^{-1}[\uparrow b]$. Hence, the pair (a, b) belongs to the enumeration. Let n be the least index such that $(a, b) = (a_n, b_n)$.

We will now find c such that the step function $\langle c; b\rangle$ belongs to Γ.

Let $i < n$. Assume that b_i and b are inconsistent. Then a_i and x must be inconsistent, since if they are consistent, then $f(a_i \sqcup x) = f(x) \in \uparrow b_i \cap \uparrow b$ contradicting that b_i and b were inconsistent. Choose $c_i \in \operatorname{approx}(x)$ such that a_i and c_i are inconsistent. Let

$$c = a \sqcup \bigsqcup_{i\in I} c_i ,$$

where I is the set of all $i < n$ such that b_i and b are inconsistent. The supremum exists since x is an upper bound. The step function $\langle c; b\rangle$ belongs to Γ. Thus, $gx = fx$. □

Proposition 3.5. *Let* (D, D^{R}, ν) *be a countably based domain representation of* X *and* $(E, E^{\mathrm{R}}, \rho, \eta)$ *be a retract domain representation of* Y*. Then any continuous function* $\varphi: X \to Y$ *can be lifted to a domain function* $g: D \to E$*.*

Proof. Define a function $f: D^{\mathrm{R}} \to E^{\mathrm{R}}$ by $f = \eta\varphi\nu$. By Lemmas 3.3 and 3.4 the function f can be extended to a function $g: D \to E$. □

4 The category DR

We introduce the category **DR** of upwards-closed retract domain representations and show that there exists a functor from a category of topological spaces into this category. We start by defining our categories.

The category **DR** is the category of upwards-closed retract domain representations [2]. Formally, the category **DR** of domain representations has as objects all tuples $(D, X, D^{\mathrm{R}}, \rho, \eta)$ where D is a domain, X is a topological space, D^{R} is an upper set of D, and $(\rho: D^{\mathrm{R}} \to X, \eta: X \to D^{\mathrm{R}})$ is a retraction-embedding pair between D^{R} and X such that if y is above some $x \in D^{\mathrm{R}}$ then $\rho y = \rho x$. Another way of expressing the requirements is to say that the space X is the retract of an upper set D^{R} where the retraction ρ is order-collapsing. By Theorem 2.3, the space X must be T_3. The retraction ρ induces an equivalence relation $\sim_D$ on D^{R} by

$$x \sim_D y \iff \rho x = \rho y \, .$$

Let $(D_1, X_1, D_1^{\mathrm{R}}, \rho_1, \eta_1)$ and $(D_2, X_2, D_2^{\mathrm{R}}, \rho_2, \eta_2)$ be domain representations. Consider the set

$$F = \left\{ f: D_1 \to D_2 \mid f[D_1^{\mathrm{R}}] \subseteq D_2^{\mathrm{R}} \text{ and } x \sim_{D_1} y \implies fx \sim_{D_2} fy \right\} \, .$$

Define an equivalence relation $\sim$ on F by

$$f \sim g \iff \forall x \in D_1^{\mathrm{R}} (fx \sim_{D_2} gx) \, .$$

A morphism of the category **DR** is an equivalence class with respect to $\sim$.

Let $\mathbf{T_3}$ be the category of all pairs $(X, \mathcal{S})$, where X is a T_3 topological space and $\mathcal{S}$ is a subbase for the topology on X. A morphism of $\mathbf{T_3}$ is a continuous function.

The forgetful functor $U: \mathbf{DR} \to \mathbf{T_3}$ is defined as follows. Let the object part of U be $(D, X, D^{\mathrm{R}}, \rho, \eta) \mapsto (X, \mathcal{B})$, where the base $\mathcal{B}$ is $\{\eta^{-1}[\uparrow a] : a \in D_c\}$. By Proposition 2.4 we have that each morphism $[\bar{f}]: D_1 \to D_2$ uniquely determines a morphism $f: X_1 \to X_2$. Let the morphism part of U be the map that takes $[\bar{f}]$ to the uniquely determined f.

4.1 The functor $R: \mathbf{T_3} \to \mathbf{DR}$

In this section we show that there exists a full and faithful functor from the category $\mathbf{T_3}$ of topological spaces to the category **DR** of domain representations.

Let $(X, \mathcal{S}) \in \mathbf{T_3}$. The family

$$P = \{\overline{S_1} \cap \cdots \cap \overline{S_n} : n < \omega, S_i \in \mathcal{S}, \overline{S_1} \cap \cdots \cap \overline{S_n} \neq \emptyset\}$$

is a *neighbourhood system* in the sense of [2]. Define $\sqsubseteq$ on P by $a \sqsubseteq b \iff b \subseteq a$. The ideal completion $D_{X,\mathcal{S}} = \mathrm{Idl}(P, \sqsubseteq)$ is a domain and $D_{X,\mathcal{S}} = (D_{X,\mathcal{S}}, X, D^{\mathrm{R}}_{X,\mathcal{S}}, \rho_{D_{X,\mathcal{S}}}, \eta_{D_{X,\mathcal{S}}})$ is a domain representation of X. (This is the construction used in the proof of Theorem 2.2). The set of representing elements is

$$D^{\mathrm{R}}_{X,\mathcal{S}} = \{I \in D_{X,\mathcal{S}} : \bigcap I = \{x\} \text{ for some } x \in X\}\,.$$

The retraction $\rho_{D_{X,\mathcal{S}}}$ is defined by

$$\rho_{D_{X,\mathcal{S}}}(I) = x \iff \bigcap I = \{x\}\,,$$

and the embedding $\eta_{D_{X,\mathcal{S}}}$ is defined by

$$\eta_{D_{X,\mathcal{S}}}(x) = \{a \in P : x \in a^\circ\}\,.$$

The representation $D_{X,\mathcal{S}}$ is, in fact, dense, i.e., $D^{\mathrm{R}}_{X,\mathcal{S}}$ is dense in $D_{X,\mathcal{S}}$.

Let R denote the map $(X, \mathcal{S}) \mapsto D_{X,\mathcal{S}}$.

Definition 4.1. Let X be a T_3 space and let $\mathcal{S}$ be a subbasis for the topology on X. The *standard domain representation* of $(X, \mathcal{S})$ is $R(X, \mathcal{S}) = D_{X,\mathcal{S}}$.

If the subbase is clear from the context we will sometimes say that we have a standard domain representation of the space X.

Given a morphism $f: X_1 \to X_2$ in $\mathbf{T_3}$ we can construct a continuous domain function $\bar{f}$, from a standard domain representation of X_1 to a standard domain representation of X_2, such that the morphism $[\bar{f}]: R(X_1, \mathcal{S}_1) \to R(X_2, \mathcal{S}_2)$ satisfies $f\rho_{X_1} = \rho_{X_2}[\bar{f}]$. We call such an $\bar{f}$ a *representation* or a *lifting* of f. The function $\bar{f}$ is defined as the extension of $\eta_2 f \rho_1 : D^{\mathrm{R}}_1 \to D^{\mathrm{R}}_2$ to D_1. The extension exists since $\eta_2 f \rho_1$ is a continuous function from a dense subset of D_1 into an injective space D_2, see [10].

Let R map a morphism f to the morphism $[\bar{f}]$.

Proposition 4.2. *The map $R: \mathbf{T_3} \to \mathbf{DR}$ is a full and faithful functor.*

Proof. The choice of base for the domain representations is irrelevant for this argument and is dropped for readability.

A function f in the equivalence class $R\,\mathrm{id}_X$ maps each element in $(RX)^{\mathrm{R}} = D^{\mathrm{R}}_X$ to an element equivalent ($\sim$) to itself. Thus, $R\,\mathrm{id}_X$ clearly acts as the identity with respect to composition, i.e., $[\mathrm{id}_{RX}] = R\,\mathrm{id}_X$.

Let $f: X_1 \to X_2$ and $g: X_2 \to X_3$. We have

$$\rho_{X_3} \circ Rf \circ Rg = f \circ \rho_{X_2} \circ Rg = f \circ g \circ \rho_{X_1} = \rho_{X_3} \circ R(f \circ g)\,,$$

that is, compositions are preserved by R.

Recall that each morphism $[\bar{f}]$ in **DR** uniquely determines a morphism f between the underlying topological spaces. Clearly, Rf must be $[\bar{f}]$. Hence, the functor R is full.

If $Rf_1 = Rf_2$ then the uniquely determined functions on the underlying spaces must be identical, i.e., $f_1 = f_2$. Hence, R is faithful. □

The functor R is not an isomorphism since not all domain representations can be obtained as RX for some X, e.g., representations that are not dense.

It is true that the space X is preserved by the composition UR. However, the base $\mathcal{B}$ obtained from the standard domain representation $R(X, \mathcal{S})$ is not necessarily the base $\mathcal{B}'$ obtained by taking all finite intersections of sets in $\mathcal{S}$. The base $\mathcal{B}$ consists of interiors of the closures of sets in $\mathcal{B}'$, which are the same only if the sets in $\mathcal{B}'$ are regular ($\overline{B}^{\circ} = B$).

The existence of a full and faithful functor is encouraging as a basis for introducing effectivity. However, the effectivity introduced would depend both on the subbasis chosen to build the domain representation and the derived basis obtained from this representation. This would probably lead to an unwieldy theory of effectivity, so we will look for an alternative way of introducing effectivity.

5 An approach to effectivity on regular spaces

Upwards-closed retract domain representations can be derived from any retraction on a domain. This construction is general in the sense that all T_3 spaces can be given representations. However, it may be the case that a particular retract domain representation cannot be reconstructed from a retraction on a domain. In particular, there might exist a non-dense representation that cannot be constructed from a retraction. For countably based domain representations we can show, using Proposition 3.5, that any coherent retract domain representation can be reconstructed from a domain retraction. Hence, we know that any effective coherent domain representation may be obtained using this construction. Although this is encouraging, one should note that the domain retraction need not be effective, even if the domain representation is effective, since Proposition 3.5 is non-effective.

5.1 Deriving domain representations from retractions

Definition 5.1. A *retraction* on a domain D is a continuous function r such that $r \circ r = r$.

Let r be a retraction on a domain D. We aim to construct upwards-closed retract domain representations of subsets of D using the retraction r and the inclusion map.

The set $\mathrm{Fix}(r)$ of all fixed points under r is clearly equal to the forward image $r[D]$. Hence, the subsets of D that get representations must in particular be subsets of $r[D]$. Let X be a subset of fixed points, and let $D^{\mathrm{R}} = r^{-1}[X]$. If for some $x \in X$ there exists a $x' \in r[D]$ such that $x \sqsubseteq x'$ then the representation cannot be upwards-closed, since x clearly is a representation of itself and x' is above x but not an approximation of x. We will therefore require the set X to consist of *maximal* fixed points.

The intended domain representation of the space X is $(D, X, D^{\mathrm{R}}, r, \iota)$. The function $r\colon D^{\mathrm{R}} \to X$ is continuous if the topology on X is weaker than the quotient topology. On the other hand, the inclusion $\iota\colon X \to D^{\mathrm{R}}$ is continuous if the topology on X is stronger than the relative topology. The topology on D^{R} is taken to be the relative topology from the Scott topology on D.

Lemma 5.2. *The quotient topology on X induced by r is weaker than the relative topology on X.*

Proof. Let U be an open set in the quotient topology on X. Then $r^{-1}[U]$ is open. For $x \in X$ we have that $x \in U$ if, and only if, $x \in r^{-1}[U]$, since x is a fixed point under r. Thus, $U = X \cap r^{-1}[U]$, which shows that U is open in the relative topology. □

The above proof does not use any information about the domain structure so the result could be stated in a more general setting. Now, somewhat surprisingly, the two topologies coincide.

Lemma 5.3. *The relative topology on X is weaker than the quotient topology on X induced by r.*

Proof. A basic open set in the relative topology is of the form $\uparrow a \cap X$ for some compact $a \in D$. Let $y \in r^{-1}[\uparrow a \cap X]$. Clearly, $ry \in \uparrow a \cap X$. By continuity of r in D there exists a compact $b \sqsubseteq y$ such that $a \sqsubseteq rb$. Thus, $y \in \uparrow b \cap D^{\mathrm{R}} \subseteq r^{-1}[\uparrow a \cap X]$. So y is in the interior of $r^{-1}[\uparrow a \cap X]$. □

Theorem 5.4. *Let D be a domain and r a retraction on D. Choose a subset X of maximal fixed points and let $D^{\mathrm{R}} = r^{-1}[X]$. Let the topology on X be the quotient topology induced by r. Then $(D, X, D^{\mathrm{R}}, r, \iota) \in$ **DR**.*

Proof. The composition $r\iota\colon X \to X$ is the identity, since the elements of X are fixed under r. Since the quotient and relative topologies coincide on X both ι and r are continuous. Hence, the retract property is satisfied.

Assume that $d \in D^{\mathrm{R}}$ and that $d \sqsubseteq d'$. By monotonicity of r, $rd \sqsubseteq rd'$. The element rd' is a fixed point of r, but rd is a maximal fixed point, hence $rd' = rd$ and $d' \in D^{\mathrm{R}} = r^{-1}[X]$. Thus, the representation is upwards closed. □

For any $a \in D_{\mathrm{c}}$, $\iota^{-1}[\uparrow a]$ is open, so $\mathcal{B} = \{\iota^{-1}[\uparrow a] : a \in D_{\mathrm{c}}\} = \{\uparrow a \cap X : a \in D_{\mathrm{c}}\}$ is a base for the topology on X.

The following result shows that it is sufficient to consider domain representations obtained in the above manner.

Proposition 5.5. *Let (D, Y, D', ρ, η) be a dense upwards-closed retract domain representation. Then there exists a retraction r on D such that the domain representation constructed from D and r, as in Theorem 5.4, represents a space homeomorphic to Y.*

Proof. Assume that $d \in D'$ and $\eta\rho d \sqsubseteq \eta\rho d'$. By upwards-closed,

$$\rho d' = \rho\eta\rho d' = \rho\eta\rho d = \rho d ,$$

and hence $\eta\rho d = \eta\rho d'$. That is, all elements in the image $\eta\rho$ are maximal (in the image of $\eta\rho$, not in D').

The composition $\eta\rho$ can be extended to a continuous function r on D since it is a continuous function from a dense subset of D into an injective space.

Let $X = \eta\rho[D']$. By the above, X is a subset of the maximal fixed points of $r = \eta\rho$. Let $D^{\mathrm{R}} = r^{-1}[X]$. By Theorem 5.4, $(D, X, D^{\mathrm{R}}, r, \iota)$ is an upwards-closed retract domain representation. Since $X = \eta[Y]$, X and Y are homeomorphic. □

Since each T_3 space X has a dense upwards-closed retract domain representation we have that there exists a domain representations of a space homeomorphic to X constructed from a retraction on a domain.

The above result also holds for all countably based coherent domains (even where D^{R} is not dense in D) by Proposition 3.5.

Definition 5.6. A domain representation $(D, X, D^{\mathrm{R}}, \rho)$ has the *closed image property* if $\rho[\uparrow a \cap D^{\mathrm{R}}]$ is closed for all $a \in D_{\mathrm{c}}$.

The closed image property implies that $\rho[\uparrow d \cap D^{\mathrm{R}}]$ is closed for any $d \in D$.

Proposition 5.7. *A retract domain representation* $(D, X, D^{\mathrm{R}}, \rho, \eta) \in \mathbf{DR}$ *has the closed image property.*

Proof. Let $x \in X$ belong to the complement of $\rho[\uparrow a \cap D^{\mathrm{R}}]$, that is $\rho^{-1}[x] \cap \uparrow a = \emptyset$. For each $\bar{x} \in \rho^{-1}[x]$ there exists a $b_{\bar{x}} \in D_{\mathrm{c}}$ such that a and $b_{\bar{x}} \sqsubseteq \bar{x}$ are inconsistent. The set

$$U = \bigcup_{\bar{x} \in \rho^{-1}[x]} \uparrow b_{\bar{x}}$$

is open. Clearly, $\eta^{-1}[U]$ and $\rho[\uparrow a \cap D^{\mathrm{R}}]$ are disjoint and x is in the open set $\eta^{-1}[U]$. Hence, the complement of $\rho[\uparrow a]$ is open, i.e., $\rho[\uparrow a]$ is closed. □

Let $(D, X, D^{\mathrm{R}}, \rho, \eta)$ be a retract domain representation. Clearly, $\eta^{-1}[\uparrow a] \subseteq \rho[\uparrow a \cap D^{\mathrm{R}}]$. However, the above result does not entail that $\rho[\uparrow a \cap D^{\mathrm{R}}]$ is the closure of $\eta^{-1}[\uparrow a]$, cf. the augmented intervals of Example 5.20.

5.2 Effective domain representations

We use the above approach of constructing domain representations from retractions to define a notion of effective domain representations.

Definition 5.8. A domain representation $(D, X, D^{\mathrm{R}}, r, \iota)$ constructed from a retraction r on D is *effective* if D is an effective domain and r is effective.

Theorem 5.9. *Let* $(D, X, D^{\mathrm{R}}, r, \iota)$ *be an effective domain representation. Then* X *is metrizable.*

Proof. The domain D is effective, so D is countably based. Since X is a retract of a countably based space, X is countably based, in fact, the base $\mathcal{B}$ is countable. The space X is T_3 by Theorem 2.3. The result now follows by Urysohn's metrization theorem. □

It also follows that the space X is normal and hence T_4.

An effective domain representation gives rise to two numberings, one of the space X, and one of a base for the topology on the space X.

For the rest of the section let $(D, X, D^{\mathrm{R}}, r, \iota)$ be an effective domain representation, where (D, α) is the effective domain. We can without loss of generality assume that α is total, i.e., $\operatorname{dom} \alpha = \omega$. There exists a canonical total numbering $\bar{\alpha}$ of the constructive subdomain D_{k} consisting of all α-computable elements of D. If $S = \alpha[W_n]$ is a consistent set, then $\bar{\alpha}n$ is defined to be the supremum of S; otherwise, $\bar{\alpha}n$ is the supremum of a consistent finite subset of S. For the detailed construction and proof, see [15, Theorem 4.4]. Let $\hat{r}$ be the recursive function tracking r with respect to $\bar{\alpha}$.

An element $x \in X$ is *computable* if there exists an $\bar{\alpha}$-index n such that $x = r\bar{\alpha}n$. Let X_{k} be the set of all computable elements of X. Define the numbering ξ of X_{k} to be the numbering $\bar{\alpha}\hat{r}$ restricted to the indices that correspond to elements in D^{R}.

In general there exists no bound on the complexity of determining if an index belongs to D^{R} since X is an arbitrary subset of maximal fixed points.

We will now look at the problem of determining if two ξ-indices represent the same element of X_{k}. Recall that $\equiv_{\bar{\alpha}}$ is Π^0_2. Compare the work by Spreen [13].

Proposition 5.10. *The relation* $\equiv_\xi$ *is* Π^0_2 *relative to* $\operatorname{dom} \xi$.

Proof. Assume that $m, n \in \operatorname{dom} \xi$. We have

$$m \equiv_\xi n \iff \xi m = \xi n \iff \bar{\alpha}\hat{r}m = \bar{\alpha}\hat{r}n \iff \hat{r}m \equiv_{\bar{\alpha}} \hat{r}n\,.$$

The result follows since $\hat{r}$ is recursive and $\equiv_{\bar{\alpha}}$ is Π^0_2. □

The following example shows that the reals have an effective representation constructed from a domain and a retraction such that $\operatorname{dom} \xi$ is Π^0_2 and equality is co-r.e. relative to $\operatorname{dom} \xi$.

Example 5.11. Let D be the interval domain with rational intervals as compact elements. Let ν be a standard numbering of the rational numbers. We note that subtraction and comparisons are computable on rational numbers with respect to the numbering ν. Let α be defined by $\alpha(\langle m, n\rangle) = [\nu m, \nu n]$, where $\operatorname{dom} \alpha = \{\langle m, n\rangle : \nu m \leq \nu n\}$. Clearly, (D, α) is an effective domain.

Define a retraction r on D_{c} by

$$r([a, b]) = \bigsqcup \{[c, d] : c < a \leq b < d\}\,,$$

and extend r to a continuous function on D. Clearly, r is an effective function. Let X be all maximal fixed points of r.

A d in D belongs to D^{R} if, and only if, for all k there exists $[a,b] \in D_{\mathrm{c}}$ such that $[a,b] \sqsubseteq d$ and $b - a \leq 2^{-k}$, so D^{R} is Π^0_2.

There exists a recursive function f that takes an $\bar{\alpha}$-index n of an element $x \in D^{\mathrm{R}}$ and a natural number k and returns an α-index of a compact element $[a,b]$ such that $\alpha f(n,k) \sqsubseteq \bar{\alpha} n$ and $b - a \leq 2^{-k}$. The compact approximations of $\bar{\alpha}$ can be enumerated uniformly in the index n. So to compute $f(n,k)$ one enumerates the compact elements until a suitable approximation is found.

Now, to decide if two ξ-indices m and n are equivalent, it is sufficient to decide if $\mathrm{Cons}(\alpha f(m,k), \alpha f(n,k))$ holds for all $k \in \omega$. Thus, $\equiv_\xi$ is co-r.e.

The base $\mathcal{B} = \{\uparrow a \cap X : a \in D_{\mathrm{c}}\}$ has a numbering β defined by

$$\beta n = \uparrow(\alpha n) \cap X\,.$$

Define a recursive relation $\prec$ on $\mathrm{dom}\,\beta$ by

$$m \prec n \iff \alpha m \sqsupseteq \alpha n\,.$$

Assume that $m \prec n$. Then $\alpha m \sqsupseteq \alpha n$ so $\uparrow(\alpha m) \subseteq \uparrow(\alpha n)$, and hence

$$\beta m = \uparrow(\alpha m) \cap X \subseteq \uparrow(\alpha n) \cap X = \beta n\,.$$

Thus, $m \prec n$ implies $\beta m \subseteq \beta n$.

Lemma 5.12. *There exists a recursive function f taking two β-indices of intersecting basic open sets and returning a β-index for the non-empty intersection.*

Proof. For any $x \in X$, $x \in \beta m \cap \beta n$ if, and only if, $\alpha m \sqsubseteq x$ and $\alpha n \sqsubseteq x$. Hence, $\alpha m \sqcup \alpha n \sqsubseteq x$. Thus, if $\beta m \cap \beta n \neq \emptyset$ then

$$\beta m \cap \beta n = \uparrow(\alpha m \sqcup \alpha n) \cap X = \beta \hat{\sqcup}(m,n)\,,$$

where $\hat{\sqcup}$ is the recursive function tracking the computable binary supremum operation on D_{c}.

Define f by

$$f(m,n) = \begin{cases} \hat{\sqcup}(m,n), & \text{if } \mathrm{Cons}(\alpha m, \alpha n); \\ \uparrow, & \text{otherwise}\,. \end{cases}$$

□

In the terminology of Spreen [14] the base $\mathcal{B}$ is a *strong basis* and this holds effectively with respect to $\prec$.

In fact, the function f of the above lemma computes an index for the intersection as soon as αm and αn are consistent. If there exists a known β-index of $\emptyset$ then intersection is computable.

Lemma 5.13. *Let k be a β-index for $\emptyset$. Then intersection is computable with respect to β.*

Proof. Define the tracking function f by

$$f(m,n) = \begin{cases} \hat{\sqcup}(m,n), & \text{if } \mathrm{Cons}(\alpha m, \alpha n); \\ k, & \text{otherwise}. \end{cases}$$

□

In general, $\equiv_\beta$ is not decidable, in particular, there exists no general way of determining if $\beta n = \emptyset$. However, for certain domain representations $\equiv_\beta$ might be decidable. This is the case for the interval domain representation of the reals discussed in Example 5.11. For the interval domain we have that a β-index n represents the empty set if, and only if, $\alpha n = [a, a]$ for some a.

Lemma 5.14. *An index of an r.e. set V can be computed, uniformly in n, such that for all $i \in \operatorname{dom} \xi$, $i \in V \iff \xi i \in \beta n$.*

Proof. For any index i of a computable $x \in X_{\mathrm{k}}$ we have

$$\begin{aligned} \xi i \in \beta n &\iff \xi i \in {\uparrow}(\alpha n) \cap X \\ &\iff \xi i \in {\uparrow}(\alpha n) \\ &\iff \alpha n \sqsubseteq \xi i \\ &\iff \alpha n \sqsubseteq r \bar{\alpha} i \\ &\iff \alpha n \sqsubseteq \bar{\alpha} \hat{r} i \,. \end{aligned}$$

The compact approximations of $\bar{\alpha}\hat{r}i$ can therefore be enumerated. The set V is obtained by enumerating all i such that αn is a compact approximation of $\bar{\alpha}\hat{r}i$. □

The above lemma states in the terminology of Spreen [14] that the numbering ξ is *computable*. Moreover, since $\mathcal{B}$ is an effectively strong basis with respect to β we have that X is an *effective* T_0 space.

Let us now look at effectivity of topological convergence with respect to our numberings.

Definition 5.15. If the r.e. set W_n enumerates β-indices of a neighbourhood base for a point $x \in X_{\mathrm{k}}$ then we say that n is a *neighbourhood index* of x.

Lemma 5.16. *Let n be a ξ-index of a point $x \in X_{\mathrm{k}}$. Then $\hat{r}n$ is a neighbourhood index of x.*

Proof. Assume that U is an open set containing x. Then there exists a basic open set $B \in \mathcal{B}$ such that $x \in B \subseteq U$. The set B is of the form ${\uparrow}a \cap X$ for some $a \sqsubseteq x$. By definition of ξ, $\hat{r}n$ is an $\bar{\alpha}$-index of x. Hence there exists an α-index $b \in W_{\hat{r}n}$ such that $a \sqsubseteq \alpha b \sqsubseteq x$. Clearly, $x \in \beta b \subseteq {\uparrow}a \cap X \subseteq U$. □

We are interested in when a converse of the above result exists, i.e., when ξ-indices can be computed from neighbourhood indices.

Definition 5.17. The numberings ξ and β *allows effective limit passing* if a ξ-index effectively can be computed from an neighbourhood index.

This is closely related to *acceptable* numberings as defined by Spreen [14] and to *admissible* numberings as defined by Kreitz and Weihrauch [11, 19]. The main difference is that while the numbering β is fixed in their settings, β is derived along with the numbering ξ in our case. That is, they investigate different numberings of the elements with respect to some numbering β of a base.

Let n be a neighbourhood index of $x \in X_{\mathrm{k}}$. For all $m \in W_n$ we have $x \in \beta m = \uparrow(\alpha m) \cap X$. If $m, m' \in W_n$ then $x \in \beta m \cap \beta m'$, and hence $\mathrm{Cons}(\alpha m, \alpha m')$. Thus, the set $\{\alpha m : m \in W_n\}$ is directed and bounded by x, so the supremum

$$d = \bigsqcup_{m \in W_n} \alpha m$$

exists and $d \sqsubseteq x$. Note that n is an $\bar{\alpha}$-index of d.

Lemma 5.18. *Let n be a neighbourhood index of $x \in X$, and let $d = \bar{\alpha} n$. If $d \in D^{\mathrm{R}}$, or, equivalently, if $n \in \mathrm{dom}\,\xi$, then $rd = x$.*

Proof. By upwards-closed and since x is a fixed point we have $rd = rx = x$. □

The following examples show two different reasons for a neighbourhood index not to be an $\bar{\alpha}$-index of an element in D^{R}, i.e., the index does not belong to $\mathrm{dom}\,\xi$.

Example 5.19. Let D be the domain with two points, bottom and top. Let the retraction be the identity. The set X consists of the top element and $D^{\mathrm{R}} = X$. This a representation of a one point space. Trivially, an r.e.-index n of the set $\{a\}$, where a is an α-index for the bottom element, is a neighbourhood index. However, $\bar{\alpha} n$ is the bottom element which does not belong to D^{R}.

It is easily checked that any other choice of retraction on the two point domain will give a numbering ξ such that every neighbourhood index is in $\mathrm{dom}\,\xi$.

Example 5.20. Let the compact elements of a domain D be the rational intervals $[a, b]$ and the augmented rational intervals $[a, b]^* = [a, b] \cup \{0\}$. The ordering on D is reverse inclusion. Define a retraction r on D_{c} by

$$r(c) = \begin{cases} \bigsqcup\{[a', b'] : a' < a, b < b'\}, & \text{if } c = [a, b]; \\ \bigsqcup\{[a', b'] : a' < \min(0, a), b' > \max(0, b)\}, & \text{if } c = [a, b]^* . \end{cases}$$

Let X consist of all maximal fixed points of r. This is simply a richer domain representation of the reals, i.e., X is homeomorphic to the reals. We will identify the reals with its homeomorphic image. Note that $\uparrow[a, b]^* \cap X = (a, b)$. Hence, an index of the set $\{[1 - \frac{1}{k}, 1 + \frac{1}{k}]^* : k \geq 1\}$ is a neighbourhood index of 1. However, $d = \bigsqcup_k [1 - \frac{1}{k}, 1 + \frac{1}{k}]^*$ represents the set $\{0, 1\}$ and hence d does not belong to D^{R}.

In fact, these two ways of failure are in a sense the only ones.

Lemma 5.21. *Let the domain representation constructed from the retraction r on D satisfy*

(i) *for all $d \in D$, if $r[\uparrow d \cap D^{\mathrm{R}}]$ is a singleton set then $d \in D^{\mathrm{R}}$, and*
(ii) *for any sequence $(a_i)_{i\in\omega}$ of compacts such that $X \cap \bigcap_i \uparrow a_i = \{x\}$ then $\bigcap_i r[\uparrow a_i \cap D^{\mathrm{R}}] = \{x\}$.*

Then a neighbourhood index n of x belongs to $\operatorname{dom}\xi$ and $\xi n = x$.

Proof. Let $d = \bar{\alpha} n$. Since n is a neighbourhood index of x we have that

$$X \cap \bigcap_{a \sqsubseteq \bar{\alpha} n} \uparrow a = \{x\} .$$

By (ii) $r[\uparrow d \cap D^{\mathrm{R}}] = \{x\}$, and by (i), $d \in D^{\mathrm{R}}$. □

Corollary 5.22. *The domain representation constructed in Example 5.11 of the reals allows effective limit passing.*

Proof. It is clear that the retraction on the interval domain satisfies the conditions of Lemma 5.21. □

We note that one of the obvious differences between Example 5.11, which allows effective limit passing, and Example 5.20, which does not allow effective limit passing, is that the former has regular sets as approximations.

6 Conclusions

In the search for a general effectivity theory to be given to T_3 spaces one problem that arises is the choice of base. It becomes visible in the competing bases that we get from the functor R of Section 4. Studying the effectivity in that setting would probably be ad hoc and confusing.

We suggest a different way of building domain representations which has the benefit that it at once gives numberings of both the base and the elements. These two numberings usually work well together, although some limitations exist. For example, the numberings do not always allow effective limit passing.

Many open questions remain. For example, does there exist a more accessible characterisation of the representations that do allow effective limit passing?

We know that the effectively represented spaces must be T_3, in fact they are metrizable. The represented spaces will be effectively Hausdorff. However, I conjecture that some spaces will fail to be effectively regular, that is, given a point in a Lacombe set, it is in general not possible to find an open set containing the point and whose closure is a subset of the Lacombe set. Is there a characterisation of the representations that will be effectively regular? Which representation are effectively metrizable?

References

1. J. Blanck. Domain representability of metric spaces. *Annals of Pure and Applied Logic*, 83:225–247, 1997.
2. J. Blanck. Domain representations of topological spaces. *Theoretical Computer Science*, 247:229–255, 2000.
3. P. di Gianantonio. Real number computability and domain theory. *Information and Computation*, 127:11–25, 1996.
4. A. Edalat. Domain theory and integration. *Theoretical Computer Science*, 151:163–193, 1995.
5. A. Edalat. Dynamical systems, measures, and fractals via domain theory. *Information and Computation*, 120:32–48, 1995.
6. A. Edalat. Power domains and iterated function systems. *Information and Computation*, 124:182–197, 1996.
7. A. Edalat. Domains for computation in mathematics, physics and exact real arithmetic. *Bulletin of Symbolic Logic*, 3(4):401–452, 1997.
8. A. Edalat and R. Heckmann. A computational model for metric spaces. *Theoretical Computer Science*, 193:53–73, 1998.
9. Y. L. Ershov. Model c of partial continuous functionals. In R. O. Gandy and J. M. E. Hyland, editors, *Logic Colloquium 76*, volume 87 of *Studies in Logic and Foundations in Mathematics*, pages 455–467. North-Holland, 1977.
10. M. H. Escardó. Injective spaces via the filter monad. *Topology Proceedings*, 22(2):97–110, 1997.
11. C. Kreitz and K. Weihrauch. Theory of representations. *Theoretical Computer Science*, 38:35–53, 1985.
12. D. Normann. The continuous functionals of finite types over the reals. Preprint, Department of Mathematics, University of Oslo, 1998.
13. D. Spreen. On some decision problems in programming. *Information and Computation*, 122(1):120–139, 1995. (Corrigendum: Inform. and Comp. 148, 241–244, 1999).
14. D. Spreen. On effective topological spaces. *The Journal of Symbolic Logic*, 63(1):185–221, 1998.
15. V. Stoltenberg-Hansen, I. Lindström, and E. R. Griffor. *Mathematical Theory of Domains*. Cambridge University Press, 1994.
16. V. Stoltenberg-Hansen and J. V. Tucker. Complete local rings as domains. *Journal of Symbolic Logic*, 53:603–624, 1988.
17. V. Stoltenberg-Hansen and J. V. Tucker. Effective algebra. In S. Abramsky et al., editors, *Handbook of Logic in Computer Science*, volume IV, pages 357–526. Oxford University Press, 1995.
18. G. A. Waagbø. *Domains-with-totality semantics for Intuitionistic Type Theory*. PhD thesis, University of Oslo, 1997.
19. K. Weihrauch. *An Introduction to Computable Analysis*. Springer, 2000.
20. K. Weihrauch and U. Schreiber. Embedding metric spaces into cpo's. *Theoretical Computer Science*, 16:5–24, 1981.